Mechanics 3

Revision Notes

April 2012

Contents

1 Further kinematics 3

Velocity, v, and displacement, x. 3

Forces which vary with speed 4

Reminder $a = \boldsymbol{v}\frac{dv}{dx}$ 4

2 Elastic strings and springs 5

Hooke's Law 5

Elastic strings 5

Elastic springs 5

Energy stored in an elastic string or spring 7

3 Impulse and work done by variable forces 9

Impulse of a variable force 9

Work done by a variable force 10

4 Newton's Law of Gravitation 11

Newton's law of gravitation 11

Connection between G and g. 11

5 Simple harmonic motion, S.H.M. 13

The basic S.H.M. equation $\ddot{x} = -\omega^2 x$ 13

$x = A \sin \omega t$ and $x = A \cos \omega t$ 13

Period and amplitude 13

$v^2 = \omega^2(a^2 - x^2)$ 13

Horizontal springs or strings 14

Vertical strings or springs 15

6 Motion in a circle 1 16

Angular velocity 16

Acceleration 16

Alternative proof 17

Motion in a horizontal circle 17

Conical pendulum 18

Banking 18

Inside an inverted vertical cone 19

7 Motion in a circle 2 20

Motion in a vertical circle 20

Proof that $a = r\omega^2$ for variable speed 20

Four types of problem 20

i Vertical motion of a particle attached to a string 21

ii Vertical motion of a particle inside a smooth sphere 22

iii Vertical motion of a particle attached to a rigid rod 23

iv Vertical motion of a particle on the outside of a smooth sphere 24

8 Centres of mass 25

Centre of mass of a lamina 25

Standard results for centre of mass of uniform laminas and arcs 28

Centres of mass of compound laminas 28

Centre of mass of a solid of revolution 29

Standard results for centre of mass of uniform bodies 30

Centres of mass of compound bodies 30

Tilting and hanging freely 31

Tilting 31

Hanging freely under gravity 31

Hemisphere in equilibrium on a slope 32

1 Further kinematics

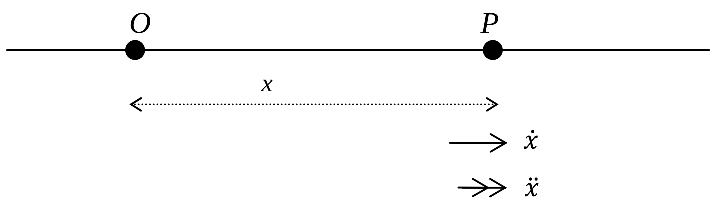
Velocity, *v*, and displacement, *x*.

We know that $v = \frac{dx}{dt} = \dot{x}$, and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$

$\Rightarrow \quad v = \int a\,dt$ and $x = \int v\,dt$

Note: $\frac{dx}{dt} = \dot{x}$ is the rate of *increase* of x, therefore it must always be measured in the direction of x increasing. For the same reason $\frac{d^2x}{dt^2} = \ddot{x}$ must also be measured in the direction of x increasing.

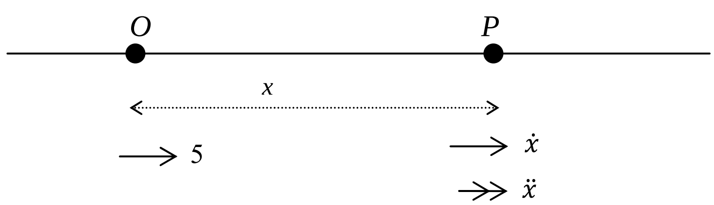
x is the displacement from O in the positive x-axis direction,



You **must** mark $\dot{x}$ and $\ddot{x}$ in the directions shown

Example: A particle moves in a straight line and passes a point, O, with speed 5 m s^{-1} at time $t = 0$. The acceleration of the particle is given by $a = 2t - 6$ m s^{-2}.
Find the distance moved in the first 6 seconds after passing O.

Solution:



$\dot{x} = v = \int \ddot{x}\;dt = \int 2t - 6\;dt = t^2 - 6t + c$; $\qquad v = 5$ when $t = 0 \Rightarrow c = 5$

$\Rightarrow \quad v = \dot{x} = t^2 - 6t + 5$

$\Rightarrow \quad x = \int \dot{x}\;dt = \int t^2 - 6t + 5\,dt = \frac{1}{3}t^3 - 3t^2 + 5t + c'$ $\qquad x = 0$ when $t = 0 \Rightarrow c' = 0$

$\Rightarrow \quad x = \frac{1}{3}t^3 - 3t^2 + 5t$.

First find when $v = 0$, $\Rightarrow t = 1$ or 5. The particle will change direction at each of these times.

$t = 0 \Rightarrow x = 0$

$t = 1 \Rightarrow x = 2\frac{1}{3}$

$t = 5 \Rightarrow x = -8\frac{1}{3}$

$t = 6 \Rightarrow x = -6$

$\Rightarrow$ particle moves forwards $2\frac{1}{3}$ from $t = 0$ to 1

particle moves backwards $10\frac{2}{3}$ from $t = 1$ to 5

particle moves forwards $2\frac{1}{3}$ from $t = 5$ to 6

$\Rightarrow$ **total** distance moved is $15\frac{1}{3}$ m.

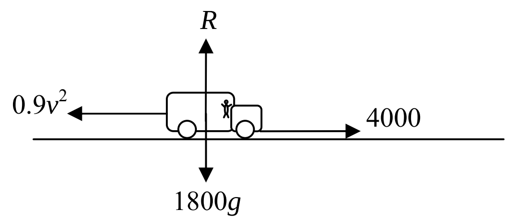
Forces which vary with speed

Reminder $\boldsymbol{a = v\frac{dv}{dx}}$

$$a = \frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx} = v\frac{dv}{dx}$$

Example: On joining a motorway a car of mass 1800 kg accelerates from 10 m s^{-1} to 30 m s^{-1}. The engine produces a constant driving force of 4000 newtons, and the resistance to motion at a speed of v m s^{-1} is $0.9v^2$ newtons. Find how far the car travels while accelerating.

Solution:



Res $\rightarrow$ $4000 - 0.9v^2 = 1800\,v\frac{dv}{dx}$

$\Rightarrow$ $\int_0^X dx = \int_{10}^{30} 1800 \times \frac{v}{4000-0.9v^2}\,dv$

$\Rightarrow$ $X = -(1800 \div 1.8) \times [\ln(4000 - 0.9v^2]_{10}^{30}$

$\Rightarrow$ $X = -1000 \times \ln\left(\frac{3190}{3910}\right) = 203.5164527$

$\Rightarrow$ the car travels a distance of 204 m, to 3 S.F.

2 Elastic strings and springs

Hooke’s Law

Elastic strings

The tension T in an elastic string is $T = \frac{\lambda x}{l}$, where l is the natural (unstretched) length of the string, x is the extension and λ is the modulus of elasticity.

T

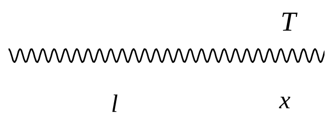
l x

When the string is slack there is no tension.

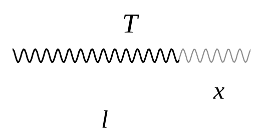
Elastic springs

The tension, or thrust, T in an elastic string is $T = \frac{\lambda x}{l}$, where l is the natural length of the spring, x is the extension, or compression, and λ is the modulus of elasticity.

In a spring there is *tension* when *stretched*, and *thrust* when *compressed*.



Tension (stretched)



Thrust (compressed)

Example: An elastic string of length 1.6 m and modulus of elasticity 30 N is stretched between two horizontal points, P and Q, which are a distance 2.4 m apart. A particle of mass m kg is then attached to the midpoint of the string, and rests in equilibrium, 0.5 m below the line PQ. Find the value of m.

Solution:

P 1.2 θ θ Q

T 0.5 T

θ θ

L

mg

By symmetry, the tensions in each half of the string will be equal.

Each half has natural length $l = 0.8$ m, and modulus of elasticity $\lambda = 30$ N.

Pythagoras $\Rightarrow$ $PL = 1.3$

$\Rightarrow$ extension in each half, x, = 0.5 m

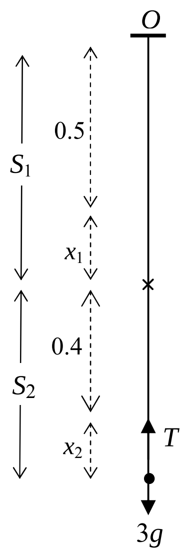
$\Rightarrow$ $T = \frac{\lambda x}{l} = \frac{30 \times 0.5}{0.8} = 18.75$

Res $\uparrow$ $2T \sin \theta = mg$ $\Rightarrow$ $2 \times 18.75 \times \frac{12}{13} = mg$

$\Rightarrow$ $m = \frac{450}{13g} = 3.532182104 = 3.5$ to 2 S.F.

Example: Two *light* strings, S_1 and S_2, are joined together at one end only. One end of the combined string is attached to the ceiling at *O*, and a mass of 3 kg is attached to the other, and allowed to hang freely in equilibrium. The moduli of S_1 and S_2 are 75 N and 120 N, and their natural lengths are 50 cm and 40 cm. Find the distance of the 3 kg mass below *O*.

Solution:



As the strings are *light*, we can ignore their masses and assume that the tensions in the two strings are equal.

(The tension is actually constant throughout the length of the combined string.)

Res ↑ $T = 3g$

For S_1,

$T = 3g = \frac{\lambda x}{l} = \frac{75x_1}{0.5} \Rightarrow x_1 = \frac{g}{50}$

For S_2,

$T = 3g = \frac{\lambda x}{l} = \frac{120x_2}{0.4} \Rightarrow x_2 = \frac{g}{100}$

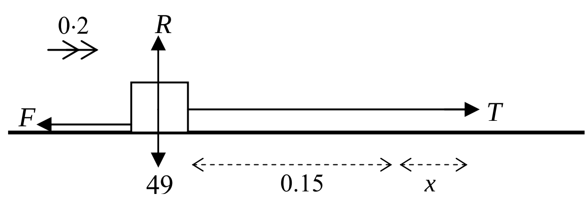
$\Rightarrow x_1 + x_2 = \frac{3g}{100} = 0.294$

$\Rightarrow$ Distance of 3 kg mass below *O*, is $0.5 + 0.4 + 0.294$

$= 1.194 = 1.2$ m to 2 S.F.

Example: A box of weight 49 N is placed on a horizontal table. It is to be pulled along by a light elastic string with natural length 15 cm and modulus of elasticity 50 N. The coefficient of friction between the box and the table is 0·4. If the acceleration of the box is 20 cm s^{-2} and the string is pulled horizontally, what is the length of the string?

Solution:



$\lambda = 50$
$\mu = 0.4$

Res ↑ $R = 49$

Box moving $\Rightarrow F = F_{max} = \mu R = 0{\cdot}4 \times 49 = 19{\cdot}6$

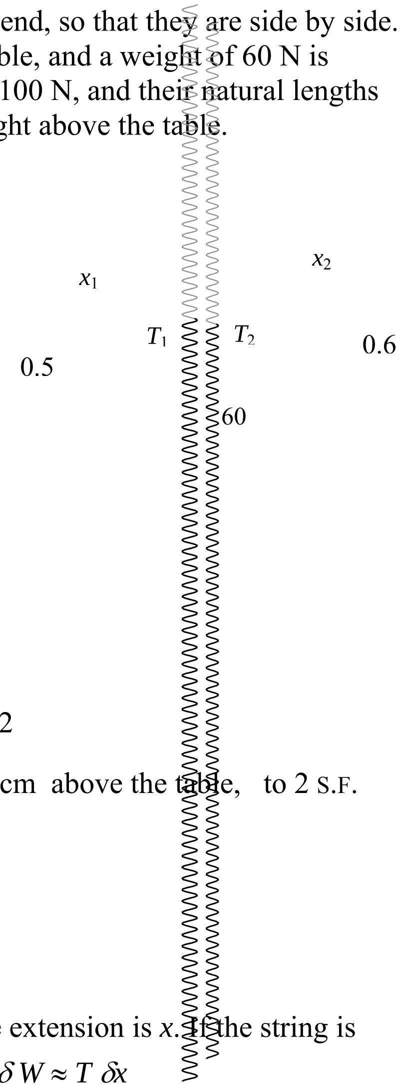
Res → N2L, $T - F = 5 \times 0{\cdot}2 \Rightarrow T = 20{\cdot}6$ $m = 49 \div 9{\cdot}8 = 5$

Hooke's Law $\Rightarrow T = \frac{50 \times x}{0{\cdot}15} = 20{\cdot}6 \Rightarrow x = 0{\cdot}0618$

$\Rightarrow$ the length of the string is $0{\cdot}15 + 0{\cdot}0618 = 0{\cdot}2118 = 0{\cdot}212$ m to 3 S.F.

Example: Two elastic springs, S_1 and S_2, are joined at each end, so that they are side by side. The bottom end of the combined spring is placed on a table, and a weight of 60 N is placed on the top. The moduli of S_1 and S_2 are 80 N and 100 N, and their natural lengths are 50 cm and 60 cm . Find the distance of the 60 N weight above the table.

Solution: $\lambda_1 = 80$, $l_1 = 0.5$, and $\lambda_2 = 100$, $l_2 = 0.6$.



The springs will have the same compressed length, but their compressions, x_1 and x_2, will differ.

Res ↑ $T_1 + T_2 = 60$ **I**

Hooke's Law $\Rightarrow$ $T_1 = \frac{80x_1}{0.5}$, $T_2 = \frac{100x_2}{0.6}$ **II**

I and **II** $\Rightarrow$ $160\,x_1 + \frac{500}{3}x_2 = 60$

and compressed lengths equal $\Rightarrow$ $0.5 - x_1 = 0.6 - x_2$

$\Rightarrow$ $x_1 = x_2 - 0.1$

$\Rightarrow$ $160(x_2 - 0.1) + \frac{500}{3}x_2 = 60$

$\Rightarrow$ $\frac{980}{3}x_2 = 76$ $\Rightarrow$ $x_2 = 0.2326530612$

$\Rightarrow$ weight is $0.6 - x_2 = 0.3673\ldots$ m $= 37$ cm above the table, to 2 S.F.

Energy stored in an elastic string or spring

For an elastic string the tension is given by $T = \frac{\lambda x}{l}$, when the extension is x. If the string is extended by a further small amount, δx, then the work done $\delta W \approx T\,\delta x$

$\Rightarrow$ Total work done in extending from $x = 0$ to $x = X$ is approximately $\sum_0^x T\,\delta x$

and, as $\delta x \to 0$, the total work done, $W = \int_0^x T\,dx = \int_0^x \frac{\lambda x}{l}\,dx$

$\Rightarrow$ $W = \frac{\lambda x^2}{2l}$ is the work done in stretching an elastic *string* from its natural length to an extension of x.

Similarly $W = \frac{\lambda x^2}{2l}$ is the work done in stretching (or compressing) an elastic *spring* from its natural length to an extension (or compression) of x.

This expression, $\frac{\lambda x^2}{2l}$, is also called the *Elastic Potential Energy*, or E.P.E., of an elastic spring or string.

Example: An elastic spring, with natural length 30 cm and modulus of elasticity 42 N, is lying on a rough horizontal table, with one end fixed to the table at *A*. The spring is held compressed so that the length of the spring is 24 cm. A teddy bear of mass 2 kg is placed on the table at the other end of the spring, and the spring is released. If the friction force is 5 N, find the speed of the teddy bear when the length of the spring is 28 cm.

Solution: At a length of 0.24 m the compression $x = 0.3 - 0.24 = 0.06$ and

the energy stored, E.P.E., is $\frac{42 \times 0.06^2}{2 \times 0.3} = 0.252$ J.

At a length of 0.28 m the compression $x = 0.3 - 0.28 = 0.02$ and

the energy stored, E.P.E., is $\frac{42 \times 0.02^2}{2 \times 0.3} = 0.028$ J,

$\Rightarrow$ energy released by the spring is $0.252 - 0.028 = 0.224$ J.

The initial speed of the teddy bear is 0, and let its final speed be v m s^{-1}.

Work done by the spring is 0.224 J, which increases the K.E.

Work done by friction is $5 \times 0.04 = 0.2$ J, which decreases the K.E.

Final K.E. = Initial K.E. + energy released by spring − work done by friction

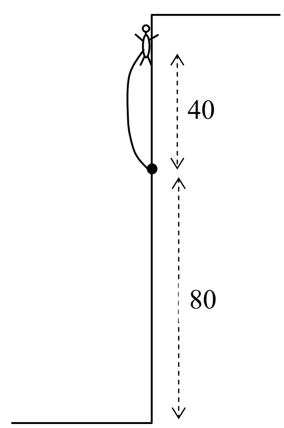
$\Rightarrow \frac{1}{2} \times 2v^2 = 0 + 0.224 - 0.2 = 0.024$

$\Rightarrow v = \sqrt{0.024} = 0.154919338$

$\Rightarrow$ speed of the teddy bear is 15 cm s^{-1}, to 2 S.F.

Example: A climber is attached to a rope of length 50 m, which is fixed to a cliff face at a point *A*, 40 metres below him. The modulus of elasticity of the rope is 9800 N, and the mass of the climber is 80 kg. The ground is 80 m below the point, *A*, to which the rope is fixed. The climber falls (oh dear!). Will he hit the ground?

Solution:



Only an idiot would consider what happens at the moment the rope becomes tight!

Assume the ground is not there – how far would he fall before being stopped by the rope. In this case both his initial and final velocities would be 0, and let the final extension of the rope be x m.

Loss in P.E. = $mgh = 80\,g \times (40 + 50 + x)$
$= 80g\,(90 + x)$, which increases K.E. and so is positive.

Work done in stretching rope, E.P.E., $= \frac{9800x^2}{2 \times 50} = 98\,x^2$

Final K.E. = Initial K.E. + Loss in P.E. − E.P.E.

$\Rightarrow 0 = 0 + 80g\,(90 + x) - 98\,x^2 \Rightarrow x^2 - 8x - 720$

$\Rightarrow x = 31.12931993$ (or negative)

The climber would fall 121.1 m, so he would hit the ground 120 m below, but not going very fast.

3 Impulse and work done by variable forces

Impulse of a variable force

A particle of mass *m* moves in a straight line under the influence of a force $F(t)$, which varies with time.

In a small time δt the impulse of the force $\delta I \approx F(t)\,\delta t$

and the total impulse from time t_1 to t_2 is $I \approx \sum_{t_1}^{t_2} F(t)\,\delta t$

and as $\delta t \to 0$, the total impulse is

$$\boldsymbol{I = \int_{t_1}^{t_2} F(t)\,dt}$$

Also, $F(t) = ma = m\frac{dv}{dt}$

$$\Rightarrow \quad \int_{t_1}^{t_2} F(t)\,dt = \int_U^V m\,dv$$

$$\Rightarrow \quad \boldsymbol{I = \int_{t_1}^{t_2} F(t)\,dt = mV - mU}$$

which is the familiar *impulse = change in momentum* equation.

Example: When a golf ball is hit, the ball is in contact with the club for 0.0008 seconds, and over that time the force is modelled by the equation $F = kt(0.0008 - t)$ newtons, where $k = 4.3 \times 10^{10}$. Taking the mass of the golf ball to be 45 grams, and modelling the ball as a particle, find the speed with which the ball leaves the club.

Soution: $F(t) = kt(0.0008 - t)$, $U = 0$, $V = ?$, $m = 0.045$

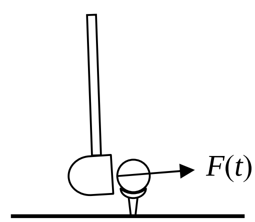
$$I = \int_0^{0.0008} F(t)\,dt = mV - mU$$

$$\Rightarrow \quad 0.045V - 0 = \int_0^{0.0008} kt(0.0008 - t)\ dt$$

$$= k\left[0.0004t^2 - \tfrac{1}{3}t^3\right]_0^{0.0008}$$

$$= 3.6693333$$

$$\Rightarrow \quad V = 81.5 \text{ m s}^{-1} \quad (\text{or } 294 \text{ km h}^{-1})$$



Work done by a variable force.

A particle of mass m moves in a straight line under the influence of a force $G(x)$, which varies with time.

Over a small distance δx the work done by the force $\delta W \approx G(x)\delta x$

and the total work done in moving from a displacement x_1 to x_2 is $W \approx \sum_{x_1}^{x_2} Gx)\,\delta x$

and as $\delta x \to 0$, the total work done is

$$\boldsymbol{W = \int_{x_1}^{x_2} G(x)\,dx}$$

Also, $G(x) = ma = m\frac{dv}{dt} = m\frac{dx}{dt} \times \frac{dv}{dx} = mv\frac{dv}{dx}$

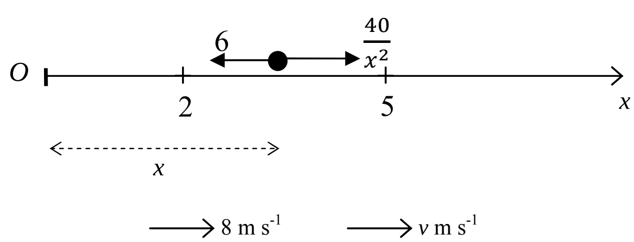
$\Rightarrow \int_{x_1}^{x_2} G(x)\,dx = \int_U^V mv\,dv$

$\Rightarrow \boldsymbol{W = \int_{x_1}^{x_2} G(x)\,dx = \frac{1}{2}mV^2 - \frac{1}{2}mU^2}$

which is the familiar *work - energy* equation.

Example: A particle of mass 0.5 kg moves on the positive x-axis under the action of a variable force $\frac{40}{x^2}$ newtons, directed away from O. The particle passes through a point 2 metres from O, with velocity 8 m s^{-1} away from O. It experiences a constant resistance force of 6 newtons. Find the speed of the particle when it is 5 metres from O.

Solution:



The work done by the resistance is $6 \times 3 = 18$ J — Decreases K.E. so negative

The work done by the force is $\int_2^5 \frac{40}{x^2}\,dx = \left[\frac{-40}{x}\right]_2^5 = 12$ J. — Increases K.E. so positive

Final K.E. = Initial K.E. − work done by resistance + work done by force

$\Rightarrow \frac{1}{2} \times 0.5V^2 = \frac{1}{2} \times 0.5 \times 8^2 - 18 + 12 = 10$ J

$\Rightarrow V = \sqrt{40}$ m s^{-1}.

4 Newton's Law of Gravitation

Tycho Brahe made many, many observations on the motion of planets. Then *Johannes Kepler*, using Brahe's results, formulated Kepler's laws of planetary motion. Finally Sir *Isaac Newton* produced his *Universal Law of Gravitation*, from which Kepler's laws could be derived.

Newton's law of gravitation

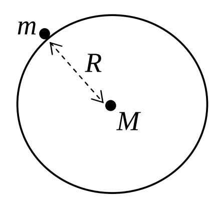
The force of attraction between two bodies of masses M_1 and M_2 is directly proportional to the product of their masses and inversely proportional to the square of the distance, d, between them:-

$$F = \frac{GM_1M_2}{d^2}$$

where G is a constant known as the *constant of gravitation.*

Connection between G and g.

It can be shown that the force of gravitation of a sphere acting on a particle lying *outside* the sphere, acts as if the whole mass of the sphere was concentrated at its centre.



Model the earth as a sphere, radius R and mass M.

The force on a particle of mass m at the surface of the earth is

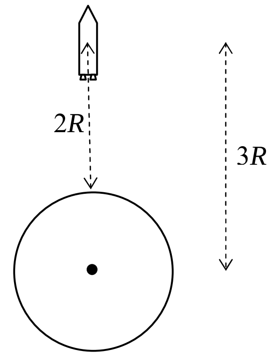
$$F = \frac{GMm}{R^2}$$

But we know that the force on m is mg, towards the centre of the earth,

$$\Rightarrow \quad \frac{GMm}{R^2} = mg \quad \Rightarrow \quad \boldsymbol{GM = gR^2}$$

This is so easy that you should work it out **every** time

Example: A space rocket is launched with speed U from the surface of the earth whose radius is R. Find, in terms of U, g and R, the speed of the rocket when it has reached a height of $2R$.



Solution: Firstly, when the rocket is at a height of $2R$, it is $3R$ from the centre of the earth.

At the surface of the earth

$$\frac{GMm}{R^2} = mg \qquad \Rightarrow \quad GM = gR^2$$

$\Rightarrow$ Gravitational force at a distance of x from the centre of the earth is $\dfrac{GMm}{x^2} = \dfrac{gR^2m}{x^2}$

$\Rightarrow$ Work done by gravity $= \int_R^{3R} \dfrac{gR^2m}{x^2}\,dx$

$$= \left[-\frac{gR^2m}{x}\right]_R^{3R} = \frac{2}{3}\,mgR$$

Decreases K.E. so negative

Final K.E. = Initial K.E. $-$ work done gravity

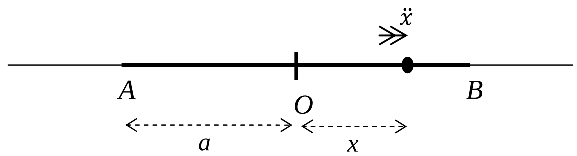
$$\Rightarrow \quad \frac{1}{2}mV^2 = \frac{1}{2}mU^2 - \frac{2}{3}mgR$$

$$\Rightarrow \quad V = \sqrt{U^2 - \frac{4}{3}gR}$$

5 Simple harmonic motion, S.H.M.

The basic S.H.M. equation $\ddot{x} = -\omega^2 x$

If a particle moves in a straight line so that its acceleration is proportional to its distance from a fixed point *O*, and directed towards *O*, then



$$\ddot{x} = -\omega^2 x$$

and the particle will oscillate between two points, *A* and *B*, with simple harmonic motion.

Notice that $\ddot{x}$ is marked in the direction of x increasing n the diagram, and, since ω^2 is positive, $\ddot{x}$ is negative, so the acceleration acts towards *O*.

$x = A \sin \omega t$ and $x = A \cos \omega t$

Solving $\ddot{x} = -\omega^2 x$, A.E. is $m^2 = -\omega^2 \quad \Rightarrow \quad m = \pm i\omega$

$\Rightarrow$ G.S. is $x = A \sin \omega t + B \cos \omega t$

If x starts from *O*, $x = 0$, then $x = a \sin \omega t$

and if x starts from *B*, $x = a$, then $x = a \cos \omega t$

Period and amplitude

From the equations $x = a \sin \omega t$ and $x = a \cos \omega t$

we can see that the *period*, the time for one complete oscillation, is $T = \dfrac{2\pi}{\omega}$,

and that the *amplitude*, maximum distance from the central point, is a.

$v^2 = \omega^2(a^2 - x^2)$

$\ddot{x} = -\omega^2 x$, and remember that $\ddot{x} = v\dfrac{dv}{dx}$

$\Rightarrow \quad v\dfrac{dv}{dx} = -\omega^2 x \quad \Rightarrow \quad \int v\,dv = \int -\omega^2 x\,dx$

$\Rightarrow \quad \frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}c$

But $v = 0$ when x is at its maximum, $x = \pm a$, $\quad \Rightarrow \quad c = a^2\omega^2$

$\Rightarrow \quad v^2 = \omega^2(a^2 - x^2)$

Example: A particle is in simple harmonic motion about *O*. When it is 6 metres from *O* its speed is 4 m s^{-1}, and its deceleration is 1.5 m s^{-2}. Find the amplitude of the oscillation, and the greatest speed as it oscillates. Find also the time taken to move a total distance of 16 m starting from the furthest point from *O*.

Solution: We are told that $v = 4$ and $\ddot{x} = -1.5$ when $x = 6$

$\ddot{x} = -\omega^2 x \quad \Rightarrow \quad -1.5 = -6\omega^2 \quad \Rightarrow \quad \omega = 0.5$ taking positive value

$v^2 = \omega^2(a^2 - x^2) \quad \Rightarrow \quad 16 = 0.5^2(a^2 - 6^2) \quad \Rightarrow \quad a = 10$ taking positive value

Starting from the furthest point from O, we use $x = a \cos \omega t = 10 \cos 0.5t$

When the particle has moved 16 metres, $x = -6$

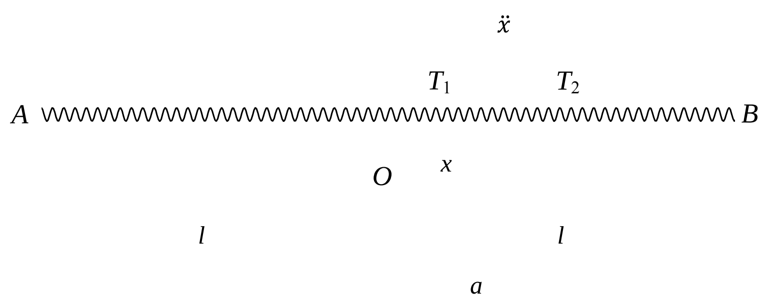
$\Rightarrow \quad -6 = 10 \cos 0.5t \quad \Rightarrow \quad t = 2 \arccos(-0.6) = 4.43$ seconds to 3 S.F.

Horizontal springs or strings

Example: Two identical springs, of natural length l and modulus λ, are joined at one end, and placed on a smooth, horizontal table. The two ends of the combined spring are fixed to two points, A and B, a distance $2l$ apart. A particle of mass m is attached to the springs at the midpoint of AB; the particle is then displaced a distance a towards B and released.

(*a*) Show that the particle moves under S.H.M.
(*b*) Find the period of the motion.
(*c*) Find the speed of the particle when it has moved through a distance of $1.5a$.

Solution: A good diagram is *essential*.



(*a*) Consider the mass at a displacement of x from O.

$T_1 = \dfrac{\lambda x}{l}$ and is a *tension*: $\quad T_2 = \dfrac{\lambda x}{l}$ and is a *thrust*

Res $\rightarrow \quad -2 \times \dfrac{\lambda x}{l} = m\ddot{x}$

$\Rightarrow \quad \ddot{x} = -\dfrac{2\lambda}{ml} x$, which is the equation of S.H.M., since $\dfrac{2\lambda}{ml} = \omega^2$ $\quad \lambda$, m and l are all positive

(Note that the diagram still works when the particle is on the left of O. x will be negative, and so both T_1 and T_2 will be negative, and will have become thrust and tension respectively.)

(b) The period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{2\lambda}}$

(c) When the particle has moved $1.5a$, it is on the left of O and $x = -0.5a$

$v^2 = \omega^2(a^2 - x^2) \quad \Rightarrow \quad v^2 = \frac{2\lambda}{ml}\left(a^2 - (-0.5a)^2\right) = \frac{3\lambda}{2ml}a^2$

$\Rightarrow \quad v = \sqrt{\frac{3\lambda}{2ml}}\, a$

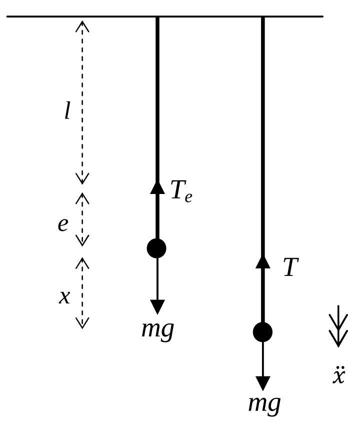
Vertical strings or springs

In these problems your diagram should show clearly

the natural length, l
the extension, e, to the equilibrium position
the extension *from* the equilibrium position, x.

Example: A mass of m hangs in equilibrium at the end of a vertical string, with natural length l and modulus λ. The mass is pulled down a further distance a and released. Show that, with certain restrictions on the value of a which you should state, the mass executes S.H.M.

Solution:



In the equilibrium position,

Res $\uparrow$ $T_e = \frac{\lambda e}{l} = mg$

At a further extension of x,

Res $\downarrow$ N2L, $mg - T = m\ddot{x}$

$\Rightarrow \quad mg - \frac{\lambda(e+x)}{l} = m\ddot{x}$

$\Rightarrow \quad \ddot{x} = -\frac{\lambda}{lm}x \qquad$ since $\frac{\lambda e}{l} = mg$

which is S.H.M., $\omega^2 = \frac{\lambda}{lm}$.

The amplitude will be a, and, since this is a *string*, the mass will perform S.H.M. only if $a \le e$.

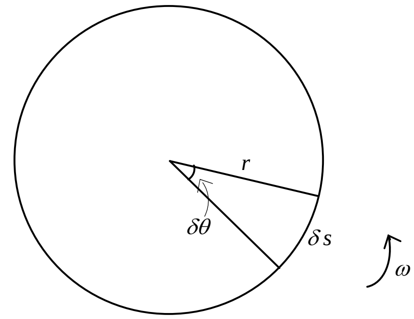
Note

- If $a > e$ the mass will perform S.H.M. as long as the string remains taut; when the string is not taut, the mass will move freely under gravity.
- If a *spring* is used then the mass will perform S.H.M. for any a (as long as the mass does not try to go above the top of the spring).

6 Motion in a circle 1

Angular velocity

A particle moves in a circle of radius r with constant speed, v.



Suppose that in a small time δt the particle moves through a small angle $\delta\theta$, then the distance moved will be $\delta s = r\,\delta\theta$

and its speed $v = \frac{\delta s}{\delta t} = r\frac{\delta\theta}{\delta t}$

and, as $\delta t \to 0$, $v = r\frac{d\theta}{dt} = r\,\dot{\theta}$

$\frac{d\theta}{dt} = \dot{\theta}$ is the angular velocity, usually written as the Greek letter omega, ω

and so, for a particle moving in a circle with radius r, its speed is $v = r\omega$

Example: Find the angular velocity of the earth, and the speed of a man standing at the equator. The equatorial radius of the earth is 6378 km.

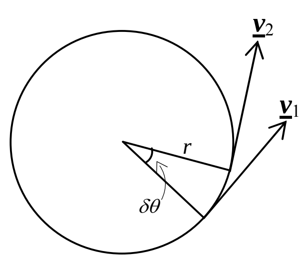
Solution: The earth rotates through an angle of 2π radians in 24 hours

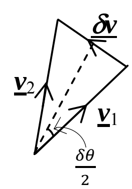
$\Rightarrow \quad \omega = \frac{2\pi}{24\times3600} = 7.272205217 \times 10^{-5} = 7.27 \times 10^{-5}$ rad s^{-1} to 3 S.F.

A man standing at the equator will be moving in a great circle

$\Rightarrow$ speed $v = r\omega = 6378000 \times 7.272205217 \times 10^{-5} = 464$ m s^{-1} to 3 S.F.

Acceleration





A particle moves in a circle of radius r with constant speed, v.

Suppose that in a small time δt the particle moves through a small angle $\delta\theta$, and that its velocity changes from $\underline{\boldsymbol{v}}_1$ to $\underline{\boldsymbol{v}}_2$,

then its change in velocity is $\underline{\boldsymbol{\delta v}} = \underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_1$, which is shown in the second diagram.

The lengths of both $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$ are v, and the angle between $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$ is $\delta\theta$. isosceles triangle

$\Rightarrow \quad \delta v = 2 \times v \sin\frac{\delta\theta}{2} \approx 2v \times \frac{\delta\theta}{2} = v\,\delta\theta$, since $\sin h \approx h$ for h small

$\Rightarrow \quad \frac{\delta v}{\delta t} \approx v\,\frac{\delta\theta}{\delta t}$

as $\delta t \to 0$, acceleration $a = \frac{dv}{dt} = v\frac{d\theta}{dt} = v\,\dot{\theta}$

But $\dot{\theta} = \omega = \frac{v}{r}$ $\Rightarrow$ $a = \frac{v^2}{r} = r\omega^2$

Notice that as $\delta\theta \to 0$, the direction of $\underline{\boldsymbol{\delta v}}$ becomes perpendicular to both $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$, and so is directed towards the centre of the circle.

The acceleration of a particle moving in a circle with speed v is $a = r\omega^2 = \frac{v^2}{r}$, and is directed towards the centre of the circle.

Alternative proof

If a particle moves, with constant speed, in a circle of radius r and centre O, then its position vector can be written

$\underline{\boldsymbol{r}} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ $\Rightarrow$ $\underline{\dot{\boldsymbol{r}}} = r\begin{pmatrix}-\sin\theta\ \dot{\theta} \\ \cos\theta\ \dot{\theta}\end{pmatrix}$ since r is constant

Particle moves with constant speed $\Rightarrow$ $\dot{\theta} = \omega$ is constant

$\Rightarrow$ $\underline{\dot{\boldsymbol{r}}} = r\omega\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$ $\Rightarrow$ speed is $v = r\omega$, and is along the tangent since $\underline{\boldsymbol{r}} \,.\, \underline{\dot{\boldsymbol{r}}} = 0$

$\Rightarrow$ $\underline{\ddot{\boldsymbol{r}}} = r\omega\begin{pmatrix}-\cos\theta\ \dot{\theta} \\ -\sin\theta\ \dot{\theta}\end{pmatrix} = -\omega^2 r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = -\omega^2\,\underline{\boldsymbol{r}}$

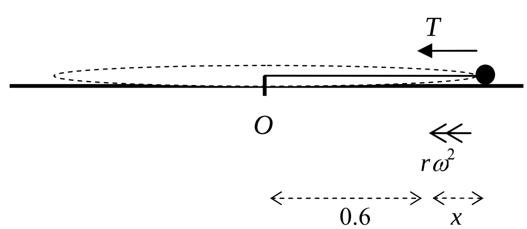
$\Rightarrow$ acceleration is $r\omega^2$ (or $\frac{v^2}{r}$) directed towards O. in opposite direction to $\underline{\boldsymbol{r}}$

Motion in a horizontal circle

Example: A blob of mass of 3 kg is describing horizontal circles on a smooth, horizontal table. The blob does 10 revolutions each minute.

An elastic string of natural length 0.6 metres and modulus 7.2 newtons is attached at one end to a fixed point O on the table. The other end is attached to the blob.

Find the full length of the string.



Solution: Let the extension of the string be x.

$\lambda = 7.2$, $l = 0.6$, $m = 3$

$\omega = \frac{10 \times 2\pi}{60} = \frac{\pi}{3}$

Res $\leftarrow$ N2L, $T = mr\omega^2 = 3(0.6 + x) \times \left(\frac{\pi}{3}\right)^2 = (0.6 + x)\frac{\pi^2}{3}$

Hooke's Law $\Rightarrow$ $T = \frac{7.2x}{0.6} = 12x$

$\Rightarrow$ $(0.6 + x)\frac{\pi^2}{3} = 12x$ $\Rightarrow$ $0.6\pi^2 + x\pi^2 = 36x$

$\Rightarrow$ $x = \frac{0.6\pi^2}{36 - \pi^2} = 0.226623537$

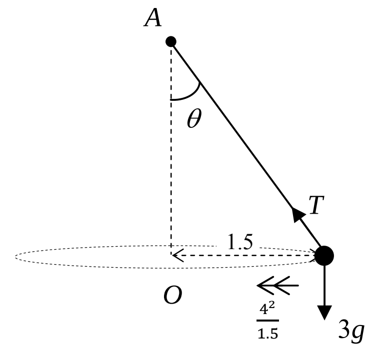
$\Rightarrow$ full length of string is $0.6 + x = 0.827$ to 3 S.F.

Conical pendulum

Example: An inextensible light string is attached at one end to a fixed point A, and at the other end to a bob of mass 3kg.

The bob is describing horizontal circles of radius 1.5 metres, with a speed of 4 m s^{-1}.

Find the angle made by the string with the downward vertical.



Solution: Acceleration $= \frac{v^2}{r} = \frac{16}{1.5} = \frac{32}{3}$

Res $\leftarrow$ N2L, $T \sin \theta = 3 \times \frac{32}{3} = 32$

Res $\uparrow$ $T \cos \theta = 3g$

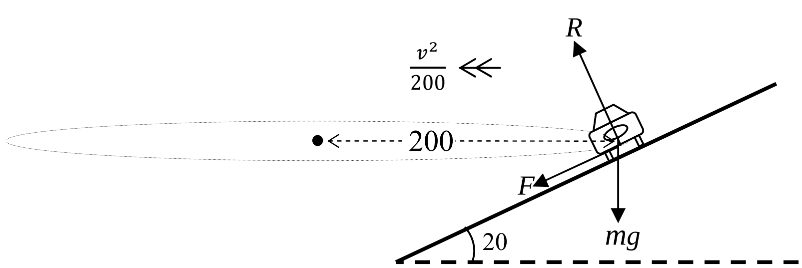
Dividing $\Rightarrow$ $\tan \theta = \frac{32}{3g} = 1.08843\ldots$

$\Rightarrow$ $\theta = 47.4^\circ$ to 1 D.P.

Banking

Example: A car is travelling round a banked curve; the radius of the curve is 200 m and the angle of banking with the horizontal is 20°. If the coefficient of friction between the tyres and the road is 0.6, find the maximum speed of the car in km h^{-1}.

Solution:



For maximum speed – (i) the friction must be acting down the slope and (ii) the friction must be at its maximum, μR.

$\Rightarrow$ $F = 0.6R$ — I

Res $\uparrow$ (perpendicular to the acceleration) $R \cos 20 = F \sin 20 + mg$ — II

Res $\leftarrow$, N2L, $F \cos 20 + R \sin 20 = m\frac{v^2}{200}$ — III

I and III $\Rightarrow$ $m\frac{v^2}{200} = R\,(0.6 \cos 20 + \sin 20)$ — IV

I and II $\Rightarrow$ $mg = R\,(\cos 20 - 0.6 \sin 20)$ — V

IV ÷ V $\Rightarrow$ $\frac{v^2}{200g} = \frac{(0.6 \cos 20 + \sin 20)}{(\cos 20 - 0.6 \sin 20)}$

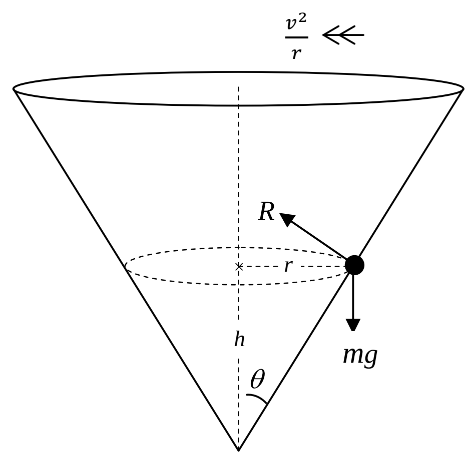
$\Rightarrow$ $v = 49.16574344$ m s^{-1} $= 176.9966764$ km h^{-1} $= 180$ km h^{-1} to 2 S.F.

Inside an inverted vertical cone

Example: A particle is describing horizontal circles on the inside of an upside down smooth cone (dunce's cap), at a height h above the vertex. Find the speed of the particle in terms of g and h.

Solution: At first, it seems as if there is not enough information. Put in letters and hope for the best!

Let the angle of the cone be 2θ, the radius of the circle in which the particle is moving r, the normal reaction R and the mass of the particle be m.



Res $\leftarrow$ N2L, $R\cos\theta = m\dfrac{v^2}{r}$

Res $\uparrow$ $\quad R\sin\theta = mg$

Dividing $\Rightarrow$ $\cot\theta = \dfrac{v^2}{rg}$

But $\quad \cot\theta = \dfrac{h}{r}$

$\Rightarrow \quad v^2 = gh \quad \Rightarrow \quad v = \sqrt{gh}$

7 Motion in a circle 2

Motion in a vertical circle

When a particle is moving under gravity in a vertical circle, the speed is no longer constant.

The 'alternative proof', given a few pages earlier, can easily be modified to show that the acceleration towards the centre is still $\frac{v^2}{r}$.

Proof that $a = \frac{v^2}{r}$ for variable speed

If a particle moves in a circle of radius r and centre O, then its position vector can be written

$$\underline{\boldsymbol{r}} = r\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\Rightarrow \quad \dot{\underline{\boldsymbol{r}}} = r\begin{pmatrix}-\sin\theta\ \dot{\theta} \\ \cos\theta\ \dot{\theta}\end{pmatrix} = r\dot{\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} \qquad \text{since } r \text{ is constant}$$

$$\Rightarrow \quad \ddot{\underline{\boldsymbol{r}}} = r\begin{pmatrix}-\cos\theta\ \dot{\theta}^2 - \sin\theta\ \ddot{\theta} \\ -\sin\theta\ \dot{\theta}^2 + \cos\theta\ \ddot{\theta}\end{pmatrix} = -r\dot{\theta}^2\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + r\ddot{\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

From this we can see that the speed is $v = r\dot{\theta} = r\omega$, and is perpendicular to the radius

We can also see that the acceleration has two components

$r\dot{\theta}^2 = r\omega^2 = \frac{v^2}{r}$ towards the centre — opposite direction to $\underline{\boldsymbol{r}}$

and $r\ddot{\theta}$ perpendicular to the radius — which is what we should expect since $v = r\dot{\theta}$, and r is constant.

In practice we shall only use $a = r\omega^2 = \frac{v^2}{r}$, directed towards the centre of the circle.

Four types of problem

1) A particle attached to an inextensible string.
2) A particle moving on the *inside* of a smooth, hollow sphere.
3) A particle attached to a rod.
4) A particle moving on the *outside* of a smooth sphere.

Types 1) and 2) are essentially the same: the particle will make complete circles as long as it is moving fast enough to keep T or $R \geq 0$,
where T is the tension in the string, or R is the normal reaction from the sphere.

Types 3) and 4) are similar when the particle is moving in the upper semi-circle, the thrust from a rod corresponds to the reaction from a sphere. *However* the particle will at some stage leave the surface of a sphere, *but* will always remain attached to a rod.
The particle will make complete circles as long as it is still moving at the top – the thrust from a rod, or reaction from a sphere, will hold it up if it is moving slowly.

Don't forget the work-energy equation – it could save you some work.

i Vertical motion of a particle attached to a string

Example: A small ball, B, of mass 500 grams hangs from a fixed point, O, by an inextensible string of length 2.5 metres. While the ball is in equilibrium it is given a horizontal impulse of magnitude 5 N s.

(*a*) Find the initial speed of the ball.

(*b*) Find the tension in the string when the string makes an angle θ with the downwards vertical.

(*c*) Find the value of θ when the string becomes slack.

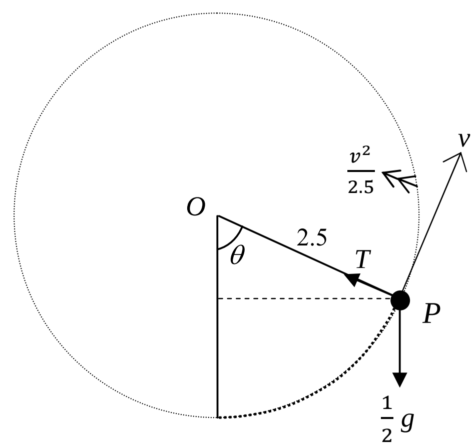
(*d*) Find the greatest height reached by the ball above the lowest point.

Solution:

(*a*) $I = mv - mu \quad \Rightarrow \quad 5 = \frac{1}{2}v \quad \Rightarrow \quad v = 10 \text{ m s}^{-1}$.

(*b*) Suppose that the particle is moving with speed v at P.

Res $\nwarrow$ N2L, $T - \frac{1}{2}g\cos\theta = \frac{1}{2}\frac{v^2}{2.5}$



From the work-energy equation

Gain in P.E. $= \frac{1}{2}g \times (2.5 - 2.5\cos\theta)$

$\frac{1}{2} \times \frac{1}{2}v^2 = \frac{1}{2} \times \frac{1}{2} \times 10^2 - \frac{1}{2}g \times 2.5(1 - \cos\theta)$

$\Rightarrow v^2 = 100 - 5g + 5g\cos\theta \qquad \ldots\ldots\ldots$ **I**

$\Rightarrow T = \frac{1}{2}g\cos\theta + \frac{1}{2}\frac{(100 - 5g + 5g\cos\theta)}{2.5}$

$\quad = \frac{1}{2}g\cos\theta + 20 - g + g\cos\theta$

$\Rightarrow T = 14.7\cos\theta + 10.2$

Notice that this still describes the situation when $\theta > 90^\circ$, *since* $\cos\theta$ *will be negative.*

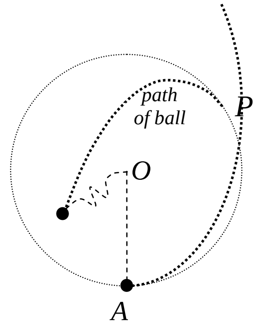
(*c*) The string will become slack when there is no tension

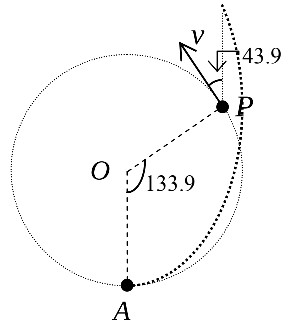
$\Rightarrow \quad T = 14.7\cos\theta + 10.2 = 0$

$\Rightarrow \quad \cos\theta = -\frac{10.2}{14.7}$

$\Rightarrow \quad \theta = 133.9378399 = 133.9^\circ$ to the nearest tenth of a degree.

(*d*)





At the greatest height, the speed will **not** be zero, so we cannot use energy to get straight to the final answer. Therefore we need to 'start again'.

We know that $v^2 = 100 - 5g + 5g\cos\theta$, from **I**, and that $\cos\theta = -\frac{10.2}{14.7}$ at P,

$\Rightarrow \quad v = \sqrt{17}$

$\Rightarrow$ initial vertical component of velocity is $u = \sqrt{17}\cos\theta$

final vertical component of velocity = 0, and $g = -9.8$

Using $v^2 = u^2 + 2as$ we get $s = 0.4497488289$

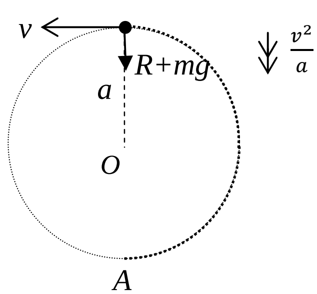
The height of P above A is $2.5 - 2.5\cos\theta = 4.234693898$

$\Rightarrow$ the greatest height of the ball above A is 4.7 m to 2 S.F.

ii Vertical motion of a particle inside a smooth sphere

Example: A particle is moving in a vertical circle inside a smooth sphere of radius a. At the lowest point of the sphere, the speed of the particle is U. What is the smallest value of U which will allow the particle to move in complete circles.

Solution: Suppose the particle is moving with speed v when it reaches the top of the sphere, and that the normal reaction of the sphere on the particle is R.



Res $\downarrow$ N2L, $\quad R + mg = m\dfrac{v^2}{a}$

For the particle to remain in contact with the sphere (i.e. to make complete circles), $R \geq 0$

$\Rightarrow \quad v^2 \geq ag$

From the lowest point, A, to the top, the gain in P.E. is $m \times g \times 2a = 2mga$

The work-energy equation gives

$\frac{1}{2}mv^2 = \frac{1}{2}mU^2 - 2mga$

$\Rightarrow \quad U^2 = v^2 + 4ga \geq 5ag \qquad$ since $v^2 \geq ag$

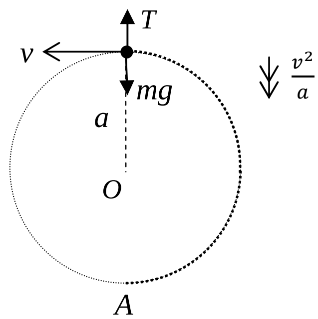
Note that if $U^2 = 5ag$ the particle will stop at the top ($v = 0$), and so **not** make complete circles $\quad \Rightarrow \quad$ For complete circles, $U > \sqrt{5ag}$.

Note that the method is **exactly the same** for a particle attached to a string, replacing the reaction, R, by the tension, T.

iii Vertical motion of a particle attached to a rigid rod

Example: A particle is attached to a rigid rod and is moving in a vertical circle of radius a. At the lowest point of the circle, the speed of the particle is U. What is the smallest value of U which will allow the particle to move in complete circles.

Solution: As long as the particle is still moving at the top of the circle, it will make complete circles.



If it is moving slowly ($v^2 < ag$), the force in the rod will be a thrust, T, and will prevent it from falling into the circle. Again, if $v = 0$, it will stop at the top,

$\Rightarrow$ for complete circles $v > 0$

From the lowest point, A, to the top
the gain in P.E. is $m \times g \times 2a = 2mga$

The work-energy equation gives

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mU^2 - 2mga$$

$\Rightarrow$ $U^2 = v^2 + 4ga > 4ag$ since $v^2 > 0$

$\Rightarrow$ For complete circles, $U > 2\sqrt{ag}$.

iv Vertical motion of a particle on the outside of a smooth sphere

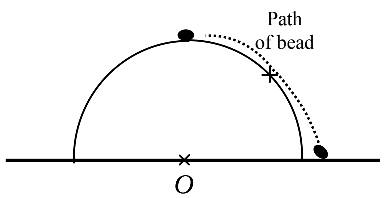
Example: A smooth hemisphere of radius a is placed on horizontal ground. A small bead of mass m is placed at the highest point and then dislodged. θ is the angle made between the line joining the centre of the hemisphere to the bead with the upward vertical.

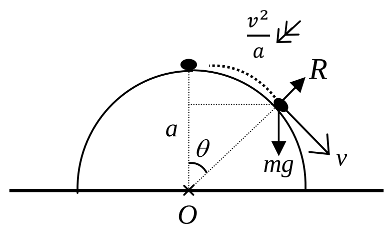
(a) Find the force of reaction between the bead and the hemisphere, in terms of m, g, a and θ.

(b) Find the value of θ when the bead leaves the surface of the hemisphere.

(c) Find the speed with which the bead strikes the ground.

Solution:





(a) When the angle is θ,

P.E. lost $= mg(a - a\cos\theta)$

Work-energy equation

$\frac{1}{2}mv^2 = 0 + mga(1 - \cos\theta)$ ………………..**I**

Res ↙ N2L, $mg\cos\theta - R = m\frac{v^2}{a}$

$\Rightarrow \quad R = mg\cos\theta - m\frac{v^2}{a}$ …………………**II**

I and **II** $\Rightarrow \quad R = mg\cos\theta - 2mg(1 - \cos\theta)$

$\Rightarrow \quad R = mg(3\cos\theta - 2)$

(b) R can never be negative, and so the bead will leave the hemisphere when $R = 0$

$\Rightarrow \quad \cos\theta = \frac{2}{3}$

$\Rightarrow \quad \theta = 48.2^{\circ}$ to the nearest tenth of a degree.

(c) The only force doing work as the particle falls from the top of the hemisphere to the ground is gravity. Note that R is always perpendicular to the path and so does no work.

P.E. lost $= mga$

Work-energy equation gives

$\frac{1}{2}mv^2 = 0 + mga$

$\Rightarrow \quad v = \sqrt{2ag}$

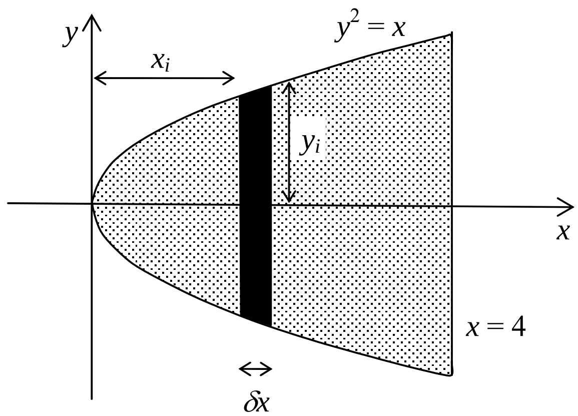
8 Centres of mass

When finding a centre of mass

1. Choose a *suitable* strip, or element.
2. Find the mass of this strip, or element – this will involve δx or δy or δz.
3. For the mass you may recognise the shape etc., or you will need $M = \sum m_i$.
   Let δx or δy or $\delta z \to 0$, and the Σ becomes an $\int$.
4. You will then need $\sum m_i x_i$. Let δx or δy or $\delta z \to 0$, and the Σ becomes an $\int$.
5. $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$ or $\frac{\sum m_i x_i}{M}$, and similarly for $\bar{y}$ and $\bar{z}$.
6. You may be able to write down the value of one or more coordinate using the symmetry of the figure.

Centre of mass of a lamina

Example: A uniform lamina is bounded by the parabola $y^2 = x$ and the line $x = 4$, and has density ρ.



By symmetry $\bar{y} = 0$.

We find the mass of the lamina, M

$$M = 2\rho \int_0^4 \sqrt{x}\ dx$$

$$= \left[\tfrac{4}{3}\rho x^{3/2}\right]_0^4 = \tfrac{32}{3}\rho$$

To find $\bar{x}$, we first choose an element with constant x co-ordinate throughout.

Take a strip parallel to the y-axis, a distance of x_i from the x-axis and width δx.

This strip is approximately a rectangle of length $2y_i$ and width δx

$\Rightarrow$ mass of typical strip $= m_i \approx 2y_i\rho\ \delta x$

$\Rightarrow \quad \sum_0^4 m_i x_i \approx \sum_0^4 2y_i \rho x_i\ \delta x$

$\quad y = \sqrt{x}$ and we let $\delta x \to 0$

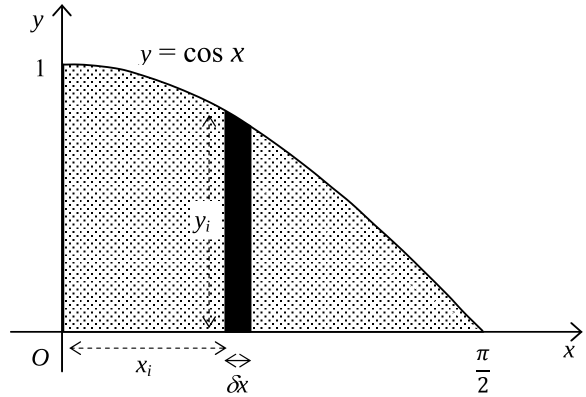
$\Rightarrow \quad \sum_0^4 2y_i\rho x_i\ \delta x \to \int_0^4 2\rho x^{3/2}\ dx = \left[\tfrac{4}{5}\rho x^{5/2}\right]_0^4 = \tfrac{128}{5}\rho$

$\Rightarrow \quad \sum m_i x_i = \tfrac{128}{5}\rho$

$$\bar{x} = \frac{\sum m_i x_i}{M} = \frac{128}{5}\rho \div \frac{32}{3}\rho = \frac{12}{5} = 2.4$$

$\Rightarrow$ centre of mass of the lamina is at $(2.4, 0)$.

Example: A uniform lamina is bounded by the x- and y-axes and the part of the curve $y = \cos x$ for which $0 \le x \le \frac{1}{2}\pi$. Find the coordinates of its centre of mass.



Solution: The figure shows the lamina and a typical strip of width δx and height $\cos x$.

1) To find the mass.

$$M = \rho \int_0^{\pi/2} \cos x \; dx$$

$$= \rho[\sin x]_0^{\pi/2} = \rho$$

2) To find $\bar{x}$

mass of typical strip $= m_i \approx y_i \rho\, \delta x$

$\Rightarrow \quad \sum_0^{\pi/2} m_i x_i \approx \sum_0^{\pi/2} y_i \rho x_i \,\delta x$

$y = \cos x$ and we let $\delta x \to 0$

$\Rightarrow \quad \sum_0^{\pi/2} m_i x_i \to \rho \int_0^{\pi/2} x \cos x \; dx = \rho\left(\frac{1}{2}\pi - 1\right)$ integrating by parts

$\Rightarrow \quad \bar{x} = \dfrac{\sum_0^{\pi/2} m_i x_i}{M} = \dfrac{\rho\left(\frac{1}{2}\pi - 1\right)}{\rho} = \frac{1}{2}\pi - 1$

3) To find $\bar{y}$ we can use the same strips, because the centre of mass of each strip is approximately $\frac{1}{2} y_i$ from the x-axis.

mass of typical strip $= m_i \approx y_i \rho\, \delta x$

$\Rightarrow \quad \sum_0^{\pi/2} m_i y_i \approx \sum_0^{\pi/2} y_i \rho \times \frac{1}{2} y_i \,\delta x$

$y = \cos x$ and we let $\delta x \to 0$

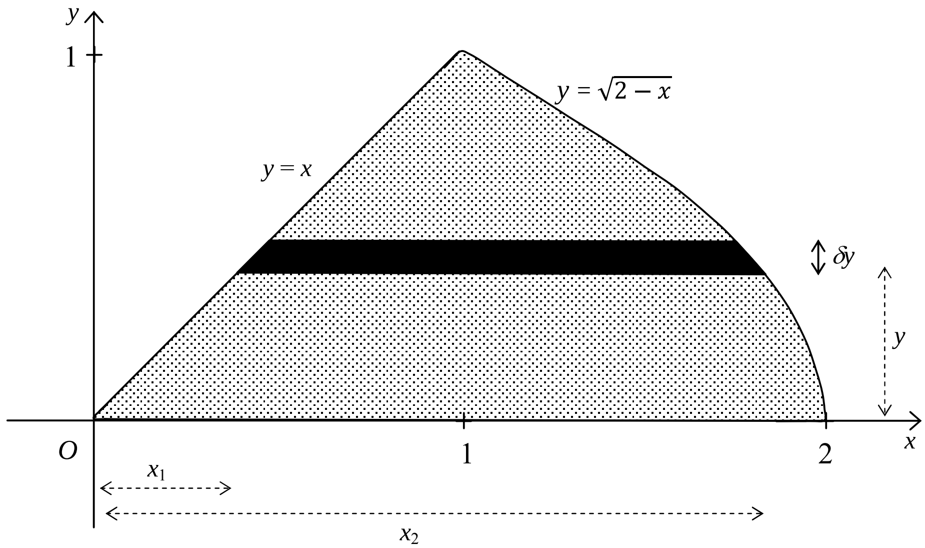
$\Rightarrow \quad \sum_0^{\pi/2} m_i y_i \to \frac{1}{2}\rho \int_0^{\pi/2} \cos^2 x \; dx = \frac{1}{8}\rho\pi$ any fool can do this integral

$\Rightarrow \quad \bar{y} = \dfrac{\sum_0^{\pi/2} m_i y_i}{M} = \dfrac{\frac{1}{8}\rho\pi}{\rho} = \frac{1}{8}\pi$

$\Rightarrow$ centre of mass is at $\left(\frac{1}{2}\pi - 1, \ \frac{1}{8}\pi\right)$

Example: A uniform lamina occupies the closed region bounded by the curve $y = \sqrt{2-x}$ and the x- and y-axes. Find the coordinates of its centre of mass.

Solution:



1) To find the mass, M.

The area = area of triangle + area under curve

$\Rightarrow \quad M = \rho\left(\frac{1}{2} \times 1 \times 1 + \int_1^2 \sqrt{2-x}\ dx\right) = \frac{7}{6}\rho$ which I am too lazy to do!

2) To find $\bar{y}$.

The typical strip is approximately a rectangle of length $x_2 - x_1$ and height δy.

The mass of the strip is $m_i = \rho\,(x_2 - x_1)\delta y$.

But $x_2 = 2 - y^2$ (lies on the curve $y = \sqrt{2-x}$), and $x_1 = y$ (lies on $y = x$)

$\Rightarrow \quad m_i = \rho\,(2 - y_i^2 - y_i)\delta y$

$\Rightarrow \quad \sum_0^1 m_i y_i \approx \sum_0^1 y_i \rho \times (2 - y_i^2 - y_i)\delta y$

As $\delta y \to 0$, $\quad \sum_0^1 m_i y_i \to \int_0^1 y\rho \times (2 - y^2 - y)\, dy = \frac{5}{12}\rho$ you ought to do this one!

$\Rightarrow \quad \bar{y} = \frac{\sum_0^1 m_i y_i}{M} = \frac{5\rho/12}{7\rho/6} = \frac{5}{14}$

3) To find $\bar{x}$.

The centre of mass of the typical strip is $\frac{1}{2}(x_2 + x_1)$ from the y-axis mid-point of strip

and $m_i = \rho\,(x_2 - x_1)\delta y$ as before

$\Rightarrow \quad \sum m_i x_i = \sum_0^1 \rho\,(x_2 - x_1)\delta y \times \frac{1}{2}(x_2 + x_1)$

But $(x_2 - x_1)(x_2 + x_1) = x_2^2 - x_1^2 = (2 - y^2)^2 - y^2 = 4 - 5y^2 + y^4$

and the limits go from 0 to 1 because of the δy.

$\Rightarrow \quad \sum m_i x_i = \sum_0^1 \frac{1}{2}\rho\,(4 - 5y^2 + y^4)\,\delta y$

As $\delta y \to 0$, $\quad \sum m_i x_i \to \int_0^1 \frac{1}{2}\rho\,(4 - 5y^2 + y^4)\, dy = \frac{19}{15}\rho$ oh, goody!

$\Rightarrow \quad \bar{x} = \frac{\sum m_i x_i}{M} = \frac{19\rho/15}{7\rho/6} = \frac{38}{35}$

$\Rightarrow$ the centre of mass is at $\left(\frac{38}{35}, \frac{5}{14}\right)$.

Standard results for centre of mass of uniform laminas and arcs

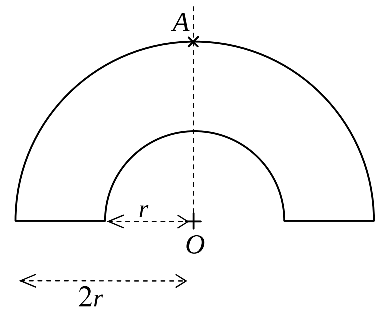
Triangle	$\frac{2}{3}$ of the way along the median, from the vertex.
Semi-circle, radius r	$\frac{4r}{3\pi}$ from centre, along axis of symmetry
Sector of circle, radius r, angle 2α	$\frac{2r\sin\alpha}{3\alpha}$ from centre, along axis of symmetry
Circular arc, radius r, angle 2α	$\frac{r\sin\alpha}{\alpha}$ from centre, along axis of symmetry

Centres of mass of compound laminas

The secret is to form a table showing the mass, or mass ratio, and position of the centre of mass for each component. Then use $\bar{x} = \frac{\sum m_i x_i}{M}$, $\bar{y} = \frac{\sum m_i y_i}{M}$ to find the centre of the compound body.

Example: A semi-circle of radius r is cut out from a uniform semi-circle of radius $2r$. Find the position of the centre of mass of the resulting shape.

Solution:



By symmetry the centre of mass will lie on the axis of symmetry, *OA*.

The mass of the compound shape is

$$M = \tfrac{1}{2}(4\pi r^2 - \pi r^2)\rho = \tfrac{3}{2}\pi r^2 \rho$$

and the centre of mass of a semi-circle is $\frac{4r}{3\pi}$ from the centre.

Lamina	compound shape	+	small semi-circle	=	large semi-circle
Mass	$\frac{3}{2}\pi r^2 \rho$		$\frac{1}{2}\pi r^2 \rho$		$2\pi r^2 \rho$
Distance from *O*	g		$\frac{4r}{3\pi}$		$\frac{8r}{3\pi}$

$$\Rightarrow \quad \tfrac{3}{2}\pi r^2 \rho \times g + \tfrac{1}{2}\pi r^2 \rho \times \frac{4r}{3\pi} = 2\pi r^2 \rho \times \frac{8r}{3\pi}$$

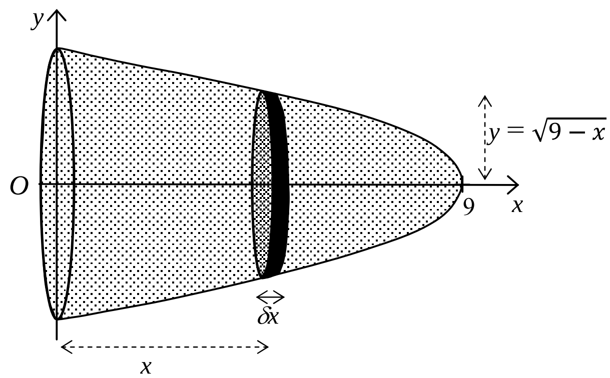
$$\Rightarrow \quad g = \frac{28}{9}r$$

The centre of mass lies on the axis of symmetry, at a distance of $\frac{28}{9}r$ from the centre.

Centre of mass of a solid of revolution

Example: A machine component has the shape of a uniform solid of revolution formed by rotating the region under the curve $y = \sqrt{9-x}$ for which $x \geq 0$ about the x-axis. Find the position of the centre of mass.

Solution:



Mass, M, of the solid $= \rho\int_0^9 \pi y^2\ dx = \rho\int_0^9 \pi(9-x)\ dx$

$\Rightarrow M = \frac{81}{2}\rho\pi.$

The diagram shows a typical thin disc of thickness δx and radius $y = \sqrt{9-x}$.

$\Rightarrow$ Mass of disc $\approx \rho\pi y^2\delta x = \rho\pi(9-x)\,\delta x$

All points in the disc have approximately the same x-coordinate

$\Rightarrow \sum m_i x_i \approx \sum_0^9 \rho\pi(9-x_i)x_i\ \delta x$

As $\delta x \to 0$, $\sum m_i x_i \to \int_0^9 \rho\pi(9-x)x\ dx = \frac{243}{2}\rho\pi$

$\Rightarrow \bar{x} = \frac{243\rho\pi/2}{81\rho\pi/2} = 3$

By symmetry, $\bar{y} = 0$

$\Rightarrow$ the centre of mass is on the x-axis, at a distance of 3 from the origin.

There are many more examples in the book, but the basic principle remains the same:

- *find the mass of the shape, M*
- *choose, carefully, a typical element, and find its mass (involving δx or δy)*
- *find* $\sum m_i x_i$ or $\sum m_i y_i$
- *let* δx *or* $\delta y \to 0$, *and find the value of the resulting integral*
- $\bar{x} = \frac{\sum m_i x_i}{M}$, $\bar{y} = \frac{\sum m_i y_i}{M}$

Standard results for centre of mass of uniform bodies

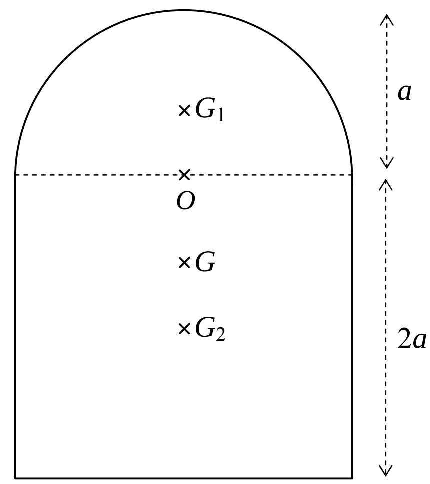
Solid hemisphere, radius r	$\frac{3r}{8}$ from centre, along axis of symmetry
Hemispherical shell, radius r	$\frac{r}{2}$ from centre, along axis of symmetry
Solid right circular cone, height h	$\frac{3h}{4}$ from vertex, along axis of symmetry
Conical shell, height h	$\frac{2h}{3}$ from vertex, along axis of symmetry

Centres of mass of compound bodies

This is very similar to the technique for compound laminas.

Example: A solid hemisphere of radius a is placed on a solid cylinder of height $2a$. Both objects are made from the same uniform material. Find the position of the centre of mass of the compound body.

Solution:



By symmetry the centre of mass will lie on the axis of symmetry.

The mass of the hemisphere is $\frac{2}{3}\pi a^3\rho$, and the mass of the cylinder is $\pi a^2 \times 2a\rho = 2\pi a^3\rho$
$\Rightarrow$ mass of the compound shape is
$M = \frac{8}{3}\pi a^3\rho$,

$OG_1 = \frac{3a}{8}$, and $OG_2 = a$

Now draw up a table

Body	hemisphere	+	cylinder	=	compound body
Mass	$\frac{2}{3}\pi a^3\rho$		$2\pi a^3\rho$		$\frac{8}{3}\pi a^3\rho$
Distance **above** O	$\frac{3a}{8}$		$-a$		$\bar{y}$

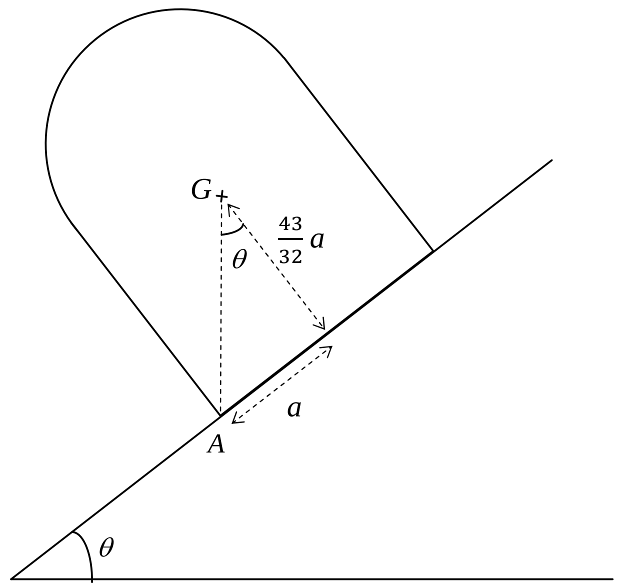
$$\Rightarrow \quad \frac{2}{3}\pi a^3\rho \times \frac{3a}{8} + 2\pi a^3\rho \times (-a) = \frac{8}{3}\pi a^3\rho \times \bar{y}$$
$$\Rightarrow \quad \bar{y} = -\frac{21}{32}a$$

$\Rightarrow$ centre of mass is at G, **below** O, where $OG = \frac{21}{32}a$, on the axis of symmetry.

Tilting and hanging freely

Tilting

Example: The compound body of the previous example is placed on a slope which makes an angle θ with the horizontal. The slope is sufficiently rough to prevent sliding. For what range of values of θ will the body remain in equilibrium.



Solution: The body will be on the point of tipping when the centre of mass, *G*, lies vertically above the lowest corner, *A*.

Centre of mass is $2a - \frac{21}{32}a$

$= \frac{43}{32}a$ from the base

At this point

$\tan\theta = \frac{a}{43a/32} = \frac{32}{43}$

$\Rightarrow \theta = 36.65610842$

The body will remain in equilibrium for

$\theta \leq 36.7^{\circ}$ to the nearest 0.1°.

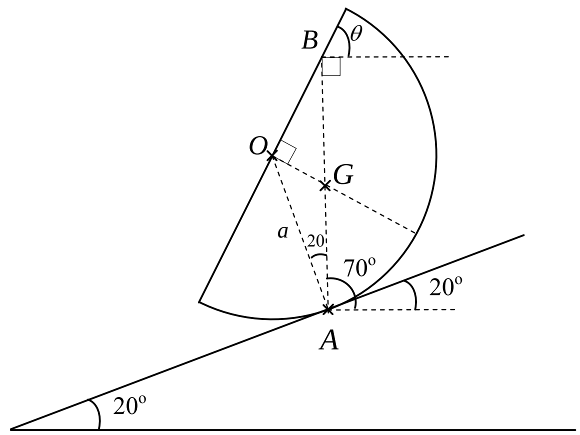
Hanging freely under gravity

This was covered in M2. For a body hanging freely from a point *A*, you should always state that AG is vertical – this is the only piece of mechanics in the question!

Hemisphere in equilibrium on a slope

Example: A uniform hemisphere rests in equilibrium on a slope which makes an angle of 20° with the horizontal. The slope is sufficiently rough to prevent the hemisphere from sliding. Find the angle made by the flat surface of the hemisphere with the horizontal.

Solution: *Don't forget the basics.*



The centre of mass, *G*, must be vertically above the point of contact, *A*. If it was not, there would be a non-zero moment about *A* and the hemisphere would not be in equilibrium.

BGA is a vertical line, so we want the angle θ.

OA must be perpendicular to the slope (radius ⊥ tangent), and with all the 90° angles around *A*, $\angle OAG = 20^{\circ}$.

Let *a* be the radius of the hemisphere

then $OG = \frac{3a}{8}$ and, using the sine rule

$\frac{\sin \angle OGA}{a} = \frac{\sin 20}{3a/8} \Rightarrow \angle OGA = 65.790....$ or $114.209...$

Clearly $\angle OGA$ is obtuse $\Rightarrow \angle OGA = 114.209...$

$\Rightarrow \angle OBG = 114.209... - 90 = 24.209...$

$\Rightarrow \theta = 90 - 24.209... = 65.8^{\circ}$ to the nearest 0.1°.

Index

Acceleration
- v dv/dx, 3
- xdv/dt, 3

Angular velocity, 16

Centres of mass
- bodies hanging freely, 31
- compound bodies, 30
- compound laminas, 28
- hemisphere on slope, 32
- laminas, 25
- solids of revolution, 29
- standard results for laminas and arcs, 28
- standard results for uniform bodies, 30
- tilting bodies, 31

Force
- impulse of variable force, 9
- varying with speed, 4

Gravitation
- link between G and g, 11
- Newton's law, 11

Hooke's Law
- elastic strings, 5
- energy stored in a string or spring, 7
- springs, 5

Impulse
- variable force, 9

Motion in a circle
- acceleration towards centre, 16
- angular velocity, 16
- banking, 18
- conical pendulum, 18
- horizontal circles, 17, 20
- inverted hollow cone, 19
- vertical circles, 20
- vertical circles at end of a rod, 23
- vertical circles at end of a string, 21
- vertical circles inside a sphere, 22
- vertical circles on outside of a sphere, 24

Simple harmonic motion
- $a \sin \omega t$ and $a \cos \omega t$, 13
- amplitude, 13
- basic equation, 13
- horizontal strings or springs, 14
- period, 13
- $v^2 = \omega^2(a^2 - x^2)$, 13
- vertical strings or springs, 15

Work
- variable force, 10