Pure FP2

Revision Notes

January 2012
### Contents

**Inequalities** ................................................................. 3
- Algebraic solutions .......................................................... 3
- Graphical solutions ........................................................... 4

**Series – Method of Differences** ........................................ 5

**Complex Numbers** ........................................................... 7
- Modulus and Argument ......................................................... 7
- Properties ........................................................................ 7
- Euler’s Relation \( e^{i\theta} \) ......................................................... 7
- Multiplying and dividing in mod-arg form ................................ 7

**De Moivre’s Theorem** .......................................................... 8
- Applications of De Moivre’s Theorem ....................................... 8

\[
z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n\theta
\] ................................................................. 8

**n^{th} roots of a complex number** ........................................ 9

**Roots of polynomial equations with real coefficients** ................. 10

**Loci on an Argand Diagram** .................................................. 10

**Transformations of the Complex Plane** ..................................... 12

**Loci and geometry** .............................................................. 13

**First Order Differential Equations** ........................................... 14
- Separating the variables, families of curves .............................. 14
- Exact Equations ................................................................. 14
- Integrating Factors .............................................................. 15
- Using substitutions ............................................................. 15

**Second Order Differential Equations** ......................................... 17
- Linear with constant coefficients ........................................... 17
  1. when \( f(x) = 0 \) ................................................................. 17
  2. when \( f(x) \neq 0 \), Particular Integrals .................................. 18

\[\frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)\] ................................................. 21
Maclaurin and Taylor Series

1) Maclaurin series
2) Taylor series
3) Taylor series – as a power series in \((x - a)\)
4) Solving differential equations using Taylor series

Standard series
Series expansions of compound functions

Polar Coordinates

Polar and Cartesian coordinates

Sketching curves
Some common curves
Areas using polar coordinates
Tangents parallel and perpendicular to the initial line
Inequalities

Algebraic solutions

Remember that if you multiply both sides of an inequality by a negative number, you must turn the inequality sign round: \(2x > 3 \Rightarrow -2x < -3\).

A difficulty occurs when multiplying both sides by, for example, \((x - 2)\); this expression is sometimes positive \((x > 2)\), sometimes negative \((x < 2)\) and sometimes zero \((x = 2)\). In this case we multiply both sides by \((x - 2)^2\), which is always positive (provided that \(x \neq 2\)).

**Example 1:** Solve the inequality \(2x + 3 < \frac{x^2}{x-2}\), \(x \neq 2\)

**Solution:** Multiply both sides by \((x - 2)^2\)

\[
\Rightarrow (2x + 3)(x - 2)^2 < x^2(x - 2)
\]

**DO NOT MULTIPLY OUT**

\[
\Rightarrow (2x + 3)(x - 2)^2 - x^2(x - 2) < 0
\]

\[
\Rightarrow (x - 2)(2x^2 - x - 6 - x^2) < 0
\]

\[
\Rightarrow (x - 2)(x - 3)(x + 2) < 0
\]

\[
\Rightarrow x < -2, \text{ or } 2 < x < 3
\]

**Note** – care is needed when the inequality is \(\leq\) or \(\geq\).

**Example 2:** Solve the inequality \(\frac{x}{x+1} \geq \frac{2}{x+3}\), \(x \neq -1, x \neq -3\)

**Solution:** Multiply both sides by \((x + 1)^2(x + 3)^2\)

which cannot be zero

\[
\Rightarrow x(x + 1)(x + 3)^2 \geq 2(x + 3)(x + 1)^2
\]

**DO NOT MULTIPLY OUT**

\[
\Rightarrow x(x + 1)(x + 3)^2 - 2(x + 3)(x + 1)^2 \geq 0
\]

\[
\Rightarrow (x + 1)(x + 3)(x^2 + 3x - 2x - 2) \geq 0
\]

\[
\Rightarrow (x + 1)(x + 3)(x + 2)(x - 1) \geq 0
\]

from sketch it looks as though the solution is

\[
x \leq -3 \quad \text{or} \quad -2 \leq x \leq -1 \quad \text{or} \quad x \geq 1
\]

BUT since \(x \neq -1, x \neq -3\),

the solution is \(x < -3 \quad \text{or} \quad -2 \leq x < -1 \quad \text{or} \quad x \geq 1\)
Graphical solutions

Example 1: On the same diagram sketch the graphs of $y = \frac{2x}{x+3}$ and $y = x - 2$.
Use your sketch to solve the inequality $\frac{2x}{x+3} \geq x - 2$

Solution: First find the points of intersection of the two graphs

$\Rightarrow \frac{2x}{x+3} = x - 2$
$\Rightarrow 2x = x^2 + x - 6$
$\Rightarrow 0 = (x - 3)(x + 2)$
$\Rightarrow x = -2$ or $3$

From the sketch we see that
$x < -3$ or $-2 \leq x \leq 3$. Note that $x \neq -3$

For inequalities involving $|2x - 5|$ etc., it is often essential to sketch the graphs first.

Example 2: Solve the inequality $|x^2 - 19| < 5(x - 1)$.

Solution: It is essential to sketch the curves first in order to see which solutions are needed.

To find the point $A$, we need to solve
$-(x^2 - 19) = 5x - 5 \Rightarrow x^2 + 5x - 24 = 0$
$\Rightarrow (x + 8)(x - 3) = 0 \Rightarrow x = -8$ or $3$

From the sketch $x \neq -8 \Rightarrow x = 3$
To find the point $B$, we need to solve

\[ + (x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 - 5x - 14 = 0 \]

\[ \Rightarrow (x - 7)(x + 2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } 7 \]

From the sketch $x \neq -2 \quad \Rightarrow \quad x = 7$

and the solution of $|x^2 - 19| < 5(x - 1)$ is $3 < x < 7$

---

**Series – Method of Differences**

The trick here is to write each line out in full and see what cancels when you add.

Do not be tempted to work each term out – you will lose the pattern which lets you cancel when adding.

*Example 1:* Write $\frac{1}{r(r+1)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n(n+1)}$.

**Solution:**

\[ \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1} \]

put $r = 1 \Rightarrow \frac{1}{1 \times 2} = \frac{1}{1} - \frac{1}{2}$

put $r = 2 \Rightarrow \frac{1}{2 \times 3} = \frac{1}{2} - \frac{1}{3}$

put $r = 3 \Rightarrow \frac{1}{3 \times 4} = \frac{1}{3} - \frac{1}{4}$

etc.

put $r = n \Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

adding $\Rightarrow \sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$
Example 2: Write \( \frac{2}{r(r+1)(r+2)} \) in partial fractions, and then use the method of differences to find the sum

\[ \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \ldots + \frac{1}{n(n+1)(n+2)}. \]

Solution:

\[ \frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2} \]

put \( r = 1 \) \( \Rightarrow \) \( \frac{2}{1\times2\times3} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3} \)

put \( r = 2 \) \( \Rightarrow \) \( \frac{2}{2\times3\times4} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \)

put \( r = 3 \) \( \Rightarrow \) \( \frac{2}{3\times4\times5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \)

put \( r = 4 \) \( \Rightarrow \) \( \frac{2}{4\times5\times6} = \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \)

\[ \vdots \]

etc.

put \( r = n-1 \) \( \Rightarrow \) \( \frac{2}{(n-1)n(n+1)} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \)

put \( r = n \) \( \Rightarrow \) \( \frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \)

adding \( \Rightarrow \) \( \sum_{1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2} \)

\[ = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2} \]

\[ = \frac{n^2+3n+2-2n-4+2n+2}{2(n+1)(n+2)} \]

\[ \Rightarrow \] \( \sum_{1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{n^2+3n}{2(n+1)(n+2)} \)

\[ \Rightarrow \] \( \sum_{1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n^2+3n}{4(n+1)(n+2)} \)
Complex Numbers

Modulus and Argument

The modulus of $z = x + iy$ is the length of $z$

$$\Rightarrow r = |z| = \sqrt{x^2 + y^2}$$

and the argument of $z$ is the angle made by $z$ with the positive $x$-axis, between $-\pi$ and $\pi$.

N.B. $\text{arg } z$ is not always equal to $\tan^{-1} \left( \frac{y}{x} \right)$

Properties

$$z = r \cos \theta + i r \sin \theta$$

$$|zw| = |z| |w|, \quad \text{ and } \quad \frac{|z|}{|w|} = \frac{|z|}{|w|}$$

$$\text{arg} (zw) = \text{arg} z + \text{arg} w, \quad \text{ and } \quad \text{arg} \left( \frac{z}{w} \right) = \text{arg} z - \text{arg} w$$

Euler’s Relation $e^{i\theta}$

$$z = e^{i\theta} = \cos \theta + i \sin \theta$$

$$\frac{1}{z} = e^{-i\theta} = \cos \theta - i \sin \theta$$

Example: Express $5e^{\left( i \frac{3\pi}{4} \right)}$ in the form $x + iy$.

Solution:

$$5e^{\left( i \frac{3\pi}{4} \right)} = 5 \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right)$$

$$= -\frac{5\sqrt{2}}{2} + i \frac{5\sqrt{2}}{2}$$

Multiplying and dividing in mod-arg form

$$re^{i\theta} \times se^{i\phi} = rs \ e^{i(\theta + \phi)}$$

$$\equiv (r \cos \theta + i \ r \sin \theta) \times (s \cos \phi + i \ s \sin \phi) = rs \cos(\theta + \phi) + i \ rs \sin(\theta + \phi)$$

and

$$re^{i\theta} + se^{i\phi} = \frac{r}{s} \ e^{i(\theta - \phi)}$$

$$\equiv (r \cos \theta + i \ r \sin \theta) \div (s \cos \phi + i \ s \sin \phi) = \frac{r}{s} \cos(\theta - \phi) + i \ \frac{r}{s} \sin(\theta - \phi)$$
De Moivre’s Theorem

\[(r e^{i\theta})^n = r^n e^{in\theta} \equiv (r \cos \theta + i r \sin \theta)^n = (r^n \cos n\theta + i r^n \sin n\theta)\]

Applications of De Moivre’s Theorem

**Example:** Express \(\sin 5\theta\) in terms of \(\sin \theta\) only.

**Solution:** From De Moivre’s Theorem we know that

\[
\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5
\]

\[
= \cos^5\theta + 5i \cos^4\theta \sin\theta + 10i^2 \cos^3\theta \sin^2\theta + 10i^3 \cos^2\theta \sin^3\theta + 5i^4 \cos\theta \sin^4\theta + i^5 \sin^5\theta
\]

Equating complex parts

\[
\Rightarrow \sin 5\theta = 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta
\]

\[
= 5(1 - \sin^2\theta)^2 \sin\theta - 10(1 - \sin^2\theta) \sin^3\theta + \sin^5\theta
\]

\[
= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta
\]

\[
z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n\theta
\]

\[
z = \cos\theta + i \sin\theta
\]

\[
\Rightarrow \quad z^n = (\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)
\]

and \(\frac{1}{z^n} = (\cos \theta - i \sin \theta)^n = (\cos n\theta - i \sin n\theta)\)

from which we can show that

\[
(z + \frac{1}{z}) = 2 \cos \theta \quad \text{and} \quad (z - \frac{1}{z}) = 2i \sin \theta
\]

\[
z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n\theta
\]

**Example:** Express \(\sin^5\theta\) in terms of \(\sin 5\theta\), \(\sin 3\theta\) and \(\sin \theta\).

**Solution:** Here we are dealing with \(\sin \theta\), so we use

\[
(2i \sin \theta)^5 = \left(z - \frac{1}{z}\right)^5
\]

\[
\Rightarrow \quad 32i \sin^5 \theta = z^5 - 5z^4 \left(\frac{1}{z}\right) + 10z^3 \left(\frac{1}{z^2}\right) - 10z^2 \left(\frac{1}{z^3}\right) + 5z \left(\frac{1}{z^4}\right) - \left(\frac{1}{z^5}\right)
\]

\[
\Rightarrow \quad 32i \sin^5 \theta = \left(z^5 - \frac{1}{z^5}\right) - 5 \left(z^3 - \frac{1}{z^3}\right) + 10 \left(z - \frac{1}{z}\right)
\]

\[
\Rightarrow \quad 32i \sin^5 \theta = 2i \sin 5\theta - 5 \times 2i \sin 3\theta + 10 \times 2i \sin \theta
\]

\[
\Rightarrow \quad \sin^5 \theta = \frac{1}{16} \left(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta\right)
\]
**n**\textsuperscript{th} roots of a complex number

The technique is the same for finding **n**\textsuperscript{th} roots of any complex number.

**Example:** Find the 4\textsuperscript{th} roots of 4 + 4i, and show the roots on an Argand Diagram.

**Solution:** We need to solve the equation \( z^4 = 4 + 4i \)

1. Let \( z = r \cos \theta + i \sin \theta \)
   \( \Rightarrow \) \( z^4 = r^4 (\cos 4\theta + i \sin 4\theta) \)
2. \(| 4 + 4i | = \sqrt{4^2 + 4^2} = \sqrt{32} \) and \( \arg (4 + 4i) = \frac{\pi}{4} \)
   \( \Rightarrow \) \( 4 + 4i = \sqrt{32} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \)
3. Then \( z^4 = 4 + 4i \) becomes \( r^4 (\cos 4\theta + i \sin 4\theta) = \sqrt{32} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \)
   \( = \sqrt{32} (\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}) \) adding 2\pi
   \( = \sqrt{32} (\cos \frac{17\pi}{4} + i \sin \frac{17\pi}{4}) \) adding 2\pi
   \( = \sqrt{32} (\cos \frac{25\pi}{4} + i \sin \frac{25\pi}{4}) \) adding 2\pi

4. \( \Rightarrow \) \( r^4 = \sqrt{32} \)
   and \( 4\theta = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \frac{25\pi}{4} \)
   \( \Rightarrow \) \( r = \sqrt[4]{32} = 1.5422 \)
   and \( \theta = \frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16}, \frac{25\pi}{16} \)

5. \( \Rightarrow \) roots are \( \sqrt[4]{32} (\cos \frac{\pi}{16} + i \sin \frac{\pi}{16}) = 1.513 + 0.301i \)
   \( \sqrt[4]{32} (\cos \frac{9\pi}{16} + i \sin \frac{9\pi}{16}) = -0.301 + 1.513i \)
   \( \sqrt[4]{32} (\cos \frac{17\pi}{16} + i \sin \frac{17\pi}{16}) = -1.513 - 0.301i \)
   \( \sqrt[4]{32} (\cos \frac{25\pi}{16} + i \sin \frac{25\pi}{16}) = 0.301 - 1.513i \)

Notice that the roots are symmetrically placed around the origin, and the angle between roots is \( \frac{2\pi}{4} = \frac{\pi}{2} \) The angle between the **n**\textsuperscript{th} roots will always be \( \frac{2\pi}{n} \).

For sixth roots the angle between roots will be \( \frac{2\pi}{6} = \frac{\pi}{3} \), and so on.
Roots of polynomial equations with real coefficients

1. **Any** polynomial equation with real coefficients,
   \[a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0, \quad \ldots \quad (I)\]
   where all \( a_i \) are real, has a complex solution

2. \( \Rightarrow \) any complex \( n \)th degree polynomial can be factorised into \( n \) linear factors over the complex numbers

3. If \( z = a + ib \) is a root of \( (I) \), then its conjugate, \( a - ib \) is also a root.

4. By pairing factors with conjugate pairs we can say that any polynomial with real coefficients can be factorised into a combination of linear and quadratic factors over the real numbers.

**Example:** Given that \( 3 - 2i \) is a root of \( z^3 - 5z^2 + 7z + 13 = 0 \)

(a) Factorise over the real numbers

(b) Find all three real roots

**Solution:**

(a) \( 3 - 2i \) is a root \( \Rightarrow 3 + 2i \) is also a root

\[ \Rightarrow (z - (3 - 2i))(z - (3 + 2i)) = (z^2 - 6z + 13) \text{ is a factor} \]

\[ \Rightarrow z^3 - 5z^2 + 7z + 13 = (z^2 - 6z + 13)(z + 1) \text{ by inspection} \]

(b) \( \Rightarrow \) roots are \( z = 3 - 2i, \ 3 + 2i \) and \(-1\)

**Loci on an Argand Diagram**

Two basic ideas

1. \( |z - w| \) is the distance from \( w \) to \( z \).

2. \( \arg(z - (1 + i)) \) is the angle made by the line joining \((1+i)\) to \( z \), with the \( x \)-axis.

**Example 1:**

\[ |z - 2 - i| = 3 \] is a circle with centre \((2 + i)\) and radius 3

**Example 2:**

\[ |z + 3 - i| = |z - 2 + i| \]

\[ \Leftrightarrow |z - (-3 + i)| = |z - (2 - i)| \]

is the locus of all points which are equidistant from the points \( A(-3, 1) \) and \( B(2, -1) \), and so is the perpendicular bisector of \( AB \).
Example 3:

\[ \arg (z - 4) = \frac{5\pi}{6} \] is a half line, from (4, 0), making an angle of \( \frac{5\pi}{6} \) with the \( x \)-axis.

Example 4:

\[ |z - 3| = 2 \mid z + 2i \mid \] is a circle (Apollonius’s circle).

To find its equation, put \( z = x + iy \)

\[ \Rightarrow \quad (x - 3)^2 + y^2 = 4(x^2 + (y + 2)^2) \]

\[ \Rightarrow \quad 3x^2 + 6x + 3y^2 + 16y + 7 = 0 \]

\[ \Rightarrow \quad (x + 1)^2 + \left( y + \frac{8}{3} \right)^2 = \frac{52}{9} \]

which is a circle with centre \( (-1, \frac{-8}{3}) \), and radius \( \frac{2\sqrt{13}}{3} \).

Example 5:

\[ \arg \left( \frac{z - 2}{z + 5} \right) = \frac{-\pi}{6} \]

\[ \Rightarrow \quad \arg(z - 2) - \arg(z + 5) = \frac{\pi}{6} \]

\[ \Rightarrow \quad \theta - \phi = \frac{\pi}{6} \]

which gives the arc of the circle as shown.

N.B.

The corresponding arc below the \( x \)-axis would have equation

\[ \arg \left( \frac{z - 2}{z + 5} \right) = -\frac{\pi}{6} \]

as \( \theta - \phi \) would be negative in this picture.
Transformations of the Complex Plane

Always start from the $z$-plane and transform to the $w$-plane, $z = x + iy$ and $w = u + iv$.

**Example 1:** Find the image of the circle $|z - 5| = 3$ under the transformation $w = \frac{1}{z-2}$.

**Solution:**

*First* rearrange to find $z$

$$w = \frac{1}{z-2} \Rightarrow z - 2 = \frac{1}{w} \Rightarrow z = \frac{1}{w} + 2$$

*Second* substitute in equation of circle

$$\Rightarrow \left| \frac{1}{w} + 2 - 5 \right| = 3 \Rightarrow \left| \frac{1-3w}{w} \right| = 3$$

$$\Rightarrow |1 - 3w| = 3|w| \Rightarrow 3\left|\frac{1}{3} - w\right| = 3|w|$$

$$\Rightarrow \left| w - \frac{1}{3} \right| = |w|$$

which is the equation of the perpendicular bisector of the line joining 0 to $\frac{1}{3}$,

$$\Rightarrow$$ the image is the line $u = \frac{1}{6}$

Always consider the ‘modulus technique’ (above) first;

if this does not work then use the $u + iv$ method shown below.

**Example 2:** Show that the image of the line $x + 4y = 4$ under the transformation $w = \frac{1}{z-3}$ is a circle, and find its centre and radius.

**Solution:**

*First* rearrange to find $z \Rightarrow z = \frac{1}{w} + 3$

The ‘modulus technique’ is not suitable here.

$$z = x + iy \quad \text{and} \quad w = u + iv$$

$$\Rightarrow \quad z = \frac{1}{w} + 3 = \frac{1}{u+iv} + 3 = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} + 3$$

$$\Rightarrow \quad x + iy = \frac{u}{u^2+v^2} + 3$$

Equating real and imaginary parts $x = \frac{u}{u^2+v^2} + 3$ and $y = \frac{-v}{u^2+v^2}$

$$\Rightarrow \quad x + 4y = 4 \quad \text{becomes} \quad \frac{u}{u^2+v^2} + 3 - \frac{4v}{u^2+v^2} = 4$$

$$\Rightarrow \quad u^2 - u + v^2 + 4v = 0$$

$$\Rightarrow \quad \left( u - \frac{1}{2} \right)^2 + (v + 2)^2 = \frac{17}{4}$$

which is a circle with centre $\left( \frac{1}{2}, -2 \right)$ and radius $\frac{\sqrt{17}}{2}$.

There are many more examples in the book, but these are the two important techniques.
Loci and geometry

It is always important to think of diagrams.

Example: \( z \) lies on the circle \(|z - 2i| = 1\).
Find the greatest and least values of \( \arg z \).

Solution: Draw a picture!

The greatest and least values of \( \arg z \)
will occur at \( B \) and \( A \).

Trigonometry tells us that
\[ \theta = \frac{\pi}{6} \]
and so greatest and least values of
\[ \arg z \] are \( \frac{2\pi}{3} \) and \( \frac{\pi}{3} \).
First Order Differential Equations

Separating the variables, families of curves

Example: Find the general solution of
\[ \frac{dy}{dx} = \frac{y}{x(x+1)}, \quad \text{for } x > 0, \]

and sketch the family of solution curves.

Solution:
\[ \frac{dy}{dx} = \frac{y}{x(x+1)} \Rightarrow \int \frac{1}{y} \, dy = \int \frac{1}{x(x+1)} \, dx = \int \frac{1}{x} - \frac{1}{x+1} \, dx \]
\[ \Rightarrow \ln y = \ln x - \ln (x+1) + \ln A \]
\[ \Rightarrow y = \frac{Ax}{x+1} = \frac{A(x+1)-1}{x+1} = A \left(1 - \frac{1}{x+1}\right) \]

Thus for varying values of \(A\) and for \(x > 0\), we have

![Graph](image)

Exact Equations

In an exact the L.H.S. is an exact derivative (really a preparation for Integrating Factors).

Example: Solve \( \sin x \, \frac{dy}{dx} + y \cos x = 3x^2 \)

Solution: Notice that the L.H.S. is an exact derivative
\[ \sin x \, \frac{dy}{dx} + y \cos x = \frac{d}{dx} (y \sin x) \]
\[ \Rightarrow \frac{d}{dx} (y \sin x) = 3x^2 \]
\[ \Rightarrow y \sin x = \int 3x^2 \, dx = x^3 + c \]
\[ \Rightarrow y = \frac{x^3 + c}{\sin x} \]
Integrating Factors

\[ \frac{dy}{dx} + Py = Q \quad \text{where } P \text{ and } Q \text{ are functions of } x \text{ only.} \]

In this case, multiply both sides by an Integrating Factor, \( R = e^{\int P \, dx} \).

The L.H.S. will now be an exact derivative, \( \frac{d}{dx} (Ry) \).

Proceed as in the above example.

Example: Solve \( x \frac{dy}{dx} + 2y = 1 \)

Solution: First divide through by \( x \)

\[ \Rightarrow \frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x} \quad \text{now in the correct form} \]

Integrating Factor, I.F., is \( R = e^{\int P \, dx} = e^{\int \frac{2}{x} \, dx} = e^{2 \ln x} = x^2 \)

\[ \Rightarrow x^2 \frac{dy}{dx} + 2xy = x \quad \text{multiplying by } x^2 \]

\[ \Rightarrow \frac{d}{dx} (x^2 y) = x, \quad \text{check that it is an exact derivative} \]

\[ \Rightarrow x^2 y = \int x \, dx = \frac{x^2}{2} + c \]

\[ \Rightarrow y = \frac{1}{2} + \frac{c}{x^2} \]

Using substitutions

Example 1: Use the substitution \( y = vx \) (where \( v \) is a function of \( x \)) to solve the equation

\[ \frac{dy}{dx} = \frac{3yx^2 + y^3}{x^3 + xy^2} \]

Solution: \( y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x \frac{dv}{dx} \)

\[ \Rightarrow \frac{dy}{dx} = \frac{3yx^2 + y^3}{x^3 + xy^2} \quad \Rightarrow \quad v + x \frac{dv}{dx} = \frac{3(vx^2 + (vx)^3)}{x^3 + x(vx)^2} = \frac{3v^3 + v^3}{1 + v^3} \]

and we can now separate the variables

\[ \Rightarrow x \frac{dv}{dx} = \frac{3v^3 + v^3}{1 + v^3} - v = \frac{3v^3 - v^3}{1 + v^3} = \frac{2v}{1 + v^3} \]

\[ \Rightarrow \frac{1 + v^2}{2v} \frac{dv}{dx} = \frac{1}{x} \]

\[ \Rightarrow \int \frac{1}{2v} + \frac{v}{2} \, dv = \int \frac{1}{x} \, dx \]

\[ \Rightarrow \frac{1}{2} \ln v + \frac{v^2}{4} = \ln x + c \]

But \( v = \frac{y}{x} \quad \Rightarrow \quad \frac{1}{2} \ln \frac{y}{x} + \frac{y^2}{4x^2} = \ln x + c \)

\[ \Rightarrow 2x^2 \ln y + y^2 = 6x^2 \ln x + c'x^2 \quad \text{\( c' \) is new arbitrary constant} \]

and I would not like to find \( y \)!!!
Example 2: Use the substitution \( y = \frac{1}{z} \) to solve the differential equation
\[
\frac{dy}{dx} = y^2 + y \cot x.
\]
Solution:
\[
y = \frac{1}{z} \Rightarrow \frac{dy}{dx} = \frac{-1}{z^2} \frac{dz}{dx}
\]
\[
\Rightarrow \frac{-1}{z^2} \frac{dz}{dx} = \frac{1}{z^2} + \frac{1}{z} \cot x
\]
\[
\Rightarrow \frac{dz}{dx} + z \cot x = -1
\]
Integrating factor is \( R = e^{\int \cot x \, dx} = e^{\ln(\sin x)} = \sin x \)
\[
\Rightarrow \sin x \frac{dz}{dx} + z \cos x = -\sin x
\]
\[
\Rightarrow \frac{d}{dx} (z \sin x) = -\sin x \quad \text{check that it is an exact derivative}
\]
\[
\Rightarrow z \sin x = \cos x + c
\]
\[
\Rightarrow z = \frac{\cos x + c}{\sin x}
\]
\[
\Rightarrow y = \frac{\sin x}{\cos x + c}
\]

Example 3: Use the substitution \( z = x + y \) to solve the differential equation
\[
\frac{dy}{dx} = \cos(x + y)
\]
Solution:
\[
z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}
\]
\[
\Rightarrow \frac{dz}{dx} = 1 + \cos z
\]
\[
\Rightarrow \int \frac{1}{1 + \cos z} \, dz = \int dx \quad \text{separating the variables}
\]
\[
\Rightarrow \int \frac{1}{2} \sec^2 \left( \frac{z}{2} \right) \, dz = x + c \quad \text{but} \quad 1 + \cos z = 1 + 2 \cos^2 \left( \frac{z}{2} \right) - 1 = 2 \cos^2 \left( \frac{z}{2} \right)
\]
\[
\Rightarrow \tan \left( \frac{z}{2} \right) = x + c
\]
But \( z = x + y \Rightarrow \tan \left( \frac{x + y}{2} \right) = x + c \)
Second Order Differential Equations

Linear with constant coefficients

\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \]

where \( a, b \) and \( c \) are constants.

(1) when \( f(x) = 0 \)

*First* write down the Auxiliary Equation, A.E

\[ am^2 + bm + c = 0 \]

and solve to find the roots \( m = \alpha \) or \( \beta \)

(i) If \( \alpha \) and \( \beta \) are both real numbers, and if \( \alpha \neq \beta \)
then the Complimentary Function, C.F., is

\[ y = A e^{\alpha x} + B e^{\beta x}, \quad \text{where } A \text{ and } B \text{ are arbitrary constants of integration} \]

(ii) If \( \alpha \) and \( \beta \) are both real numbers, and if \( \alpha = \beta \)
then the Complimentary Function, C.F., is

\[ y = (A + Bx) e^{\alpha x}, \quad \text{where } A \text{ and } B \text{ are arbitrary constants of integration} \]

(iii) If \( \alpha \) and \( \beta \) are both complex numbers, and if \( \alpha = a + ib, \beta = a - ib \)
then the Complimentary Function, C.F.,

\[ y = e^{\alpha x}(A \sin bx + B \cos bx), \quad \text{where } A \text{ and } B \text{ are arbitrary constants of integration} \]

Example 1: Solve \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 3 = 0 \)

Solution: A.E. is \( m^2 + 2m - 3 = 0 \)

\[ (m - 1)(m + 3) = 0 \]

\[ m = 1 \text{ or } -3 \]

\[ y = Ae^x + Be^{-3x} \quad \text{when } f(x) = 0, \text{ the C.F. is the solution} \]

Example 2: Solve \( \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 9 = 0 \)

Solution: A.E. is \( m^2 + 6m + 9 = 0 \)

\[ (m + 3)^2 = 0 \]

\[ m = -3 \text{ (and } -3) \quad \text{repeated root} \]

\[ y = (A + Bx)e^{-3x} \quad \text{when } f(x) = 0, \text{ the C.F. is the solution} \]
Example 3: Solve \( \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 13 = 0 \)

Solution:
A.E. is \( m^2 + 4m + 13 = 0 \)
\[ \Rightarrow (m + 2)^2 - (3i)^2 = 0 \]
\[ \Rightarrow (m + 2 + 3i)(m + 2 - 3i) = 0 \]
\[ \Rightarrow m = -2 - 3i \text{ or } -2 + 3i \]
\[ \Rightarrow y = e^{-2x}(A \sin 3x + B \cos 3x) \]
when \( f(x) = 0 \), the C.F. is the solution

(2) when \( f(x) \neq 0 \), Particular Integrals

First proceed as in (1) to find the Complimentary Function, then use the rules below to find a Particular Integral, P.I.

Second the General Solution, G.S., is found by adding the C.F. and the P.I.

\[ \Rightarrow \text{G.S.} = \text{C.F.} + \text{P.I.} \]

Note that it does not matter what P.I. you use, so you might as well find the easiest, which is what these rules do.

(1) \( f(x) = e^{kx} \).
Try \( y = Ae^{kx} \)
unless \( e^{kx} \) appears in the C.F., in which case try \( y = Cxe^{kx} \)
unless \( xe^{kx} \) appears in the C.F., in which case try \( y = Cx^2e^{kx} \).

(2) \( f(x) = \sin kx \) or \( f(x) = \cos kx \)
Try \( y = C \sin kx + D \cos kx \)
unless \( \sin kx \) or \( \cos kx \) appear in the C.F., in which case
try \( y = x(C \sin kx + D \cos kx) \).

(3) \( f(x) = \text{a polynomial of degree } n. \)
Try \( f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 \)
unless a number, on its own, appears in the C.F., in which case
try \( f(x) = x(a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0) \).

(4) In general
to find a P.I., try something like \( f(x) \), unless this appears in the C.F. (or if there is a problem), then try something like \( xf(x) \).
Example 1: Solve \( \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} + 5y = 2x \)

Solution: A.E. is \( m^2 + 6m + 5 = 0 \)  
\[ \Rightarrow (m + 5)(m + 1) = 0 \quad \Rightarrow \quad m = -5 \text{ or } -1 \]
\( \Rightarrow \) C.F. is \( y = Ae^{-5x} + Be^{-x} \)

For the P.I., try \( y = Cx + D \)  
\( \Rightarrow \) \( \frac{dy}{dx} = C \) and \( \frac{d^2y}{dx^2} = 0 \)

Substituting in the differential equation gives  
\[ 0 + 6C + 5(Cx + D) = 2x \]
\( \Rightarrow \) \( 5C = 2 \) comparing coefficients of \( x \)
\( \Rightarrow \) \( C = \frac{2}{5} \)
and \( 6C + 5D = 0 \) comparing constant terms  
\( \Rightarrow \) \( D = -\frac{12}{25} \)
\( \Rightarrow \) P.I. is \( y = \frac{2}{5}x - \frac{12}{25} \)
\( \Rightarrow \) G.S. is \( y = Ae^{-5x} + Be^{-x} + \frac{2}{5}x - \frac{12}{25} \)

Example 2: Solve \( \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = e^{3x} \)

Solution: A.E. is \( m^2 - 6m + 9 = 0 \)  
\( \Rightarrow \) \( (m - 3)^2 = 0 \)
\( \Rightarrow \) \( m = 3 \) repeated root  
\( \Rightarrow \) C.F. is \( y = (Ax + B)e^{3x} \)

In this case, both \( e^{3x} \) and \( xe^{3x} \) appear in the C.F., so for a P.I. we try \( y = Cx^2e^{3x} \)  
\( \Rightarrow \) \( \frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x} \)
and \( \frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cx^2e^{3x} + 9Cx^2e^{3x} \)

Substituting in the differential equation gives  
\[ 2Ce^{3x} + 12Cxe^{3x} + 9Cx^2e^{3x} - 6(2Cxe^{3x} + 3Cx^2e^{3x}) + 9Cx^2e^{3x} = e^{3x} \]
\( \Rightarrow \) \( 2Ce^{3x} = e^{3x} \)
\( \Rightarrow \) \( C = \frac{1}{2} \)
\( \Rightarrow \) P.I. is \( y = \frac{1}{2}x^2e^{3x} \)
\( \Rightarrow \) G.S. is \( y = (Ax + B)e^{3x} + \frac{1}{2}x^2e^{3x} \)
Example 3: Solve $\frac{d^2x}{dt^2} - x = 4 \cos 2t$
given that $x = 0$ and $\dot{x} = 1$ when $t = 0$.

Solution: A.E. is $m^2 - 1 = 0$
$\Rightarrow$ $m = \pm 1$
$\Rightarrow$ C.F. is $x = Ae^t + Be^{-t}$

For the P.I. try $x = C \sin 2t + D \cos 2t$
$\Rightarrow$ $\dot{x} = 2C \cos 2t - 2D \sin 2t$
and $\ddot{x} = -4C \sin 2t - 4D \cos 2t$

Substituting in the differential equation gives

$(-4C \sin 2t - 4D \cos 2t) - (C \sin 2t + D \cos 2t) = 4 \cos 2t$
$\Rightarrow$ $-5C = 0$ \hspace{1cm} \text{comparing coefficients of } \sin 2t$
and $-5D = 4$ \hspace{1cm} \text{comparing coefficients of } \cos 2t$
$\Rightarrow$ $C = 0$ \hspace{0.5cm} and \hspace{0.5cm} $D = \frac{-5}{4}$
$\Rightarrow$ P.I. is $x = \frac{-5}{4} \cos 2t$
$\Rightarrow$ G.S. is $x = Ae^t + Be^{-t} - \frac{5}{4} \cos 2t$
$\Rightarrow$ $\dot{x} = Ae^t - Be^{-t} + \frac{5}{2} \sin 2t$

$x = 0$ and when $t = 0$ \hspace{1cm} $\Rightarrow$ $0 = A + B - \frac{5}{4}$
and $\dot{x} = 1$ when $t = 0$ \hspace{1cm} $\Rightarrow$ $1 = A - B$
$\Rightarrow$ $A = \frac{9}{8}$ \hspace{0.5cm} and \hspace{0.5cm} $B = \frac{1}{8}$
$\Rightarrow$ solution is $x = \frac{9}{8} e^t + \frac{1}{8} e^{-t} - \frac{5}{4} \cos 2t$
D.E.s of the form \( ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = f(x) \)

Substitute \( x = e^u \)

\[
\Rightarrow \quad \frac{dx}{du} = e^u = x
\]

and \( \frac{dy}{du} = \frac{dx}{du} \times \frac{dy}{dx} \Rightarrow \frac{dy}{du} = x \frac{dy}{dx} \) result I

But \( \frac{d^2y}{du^2} = \frac{d}{du} \left( \frac{dy}{du} \right) = \frac{d^2y}{dx^2} \times \frac{dx}{du} \times \frac{dx}{du} \times \frac{dx}{du} \times \frac{dx}{du} \) using the chain rule

\[
= \frac{d}{dx} \left( \frac{dy}{dx} \right) \frac{dx}{du} \times \frac{dx}{du} \times \frac{dx}{du} \times \frac{dx}{du} \times \frac{dx}{du} \text{ product rule}
\]

\[
\Rightarrow \quad \frac{d^2y}{du^2} = x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} \quad \text{since} \quad \frac{dx}{du} = x
\]

\[
\Rightarrow \quad x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du} \quad \text{and} \quad x \frac{dy}{dx} = \frac{dy}{du}
\]

Thus we have \( x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du} \) and \( x \frac{dy}{dx} = \frac{dy}{du} \)

substituting these in the original equation leads to a second order D.E. with constant coefficients.

**Example:** Solve the differential equation \( x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = -2x^2 \).

**Solution:** Using the substitution \( x = e^u \), and proceeding as above

\[
\Rightarrow \quad \frac{d^2y}{du^2} - \frac{dy}{du} - 3 \frac{dy}{du} + 3y = -2e^{2u}
\]

\[
\Rightarrow \quad \frac{d^2y}{du^2} - 4 \frac{dy}{du} + 3y = -2e^{2u}
\]

A.E. is \( m^2 - 4m + 3 = 0 \)

\( (m-3)(m-1) = 0 \) \( \Rightarrow \) \( m = 3 \) or \( 1 \)

C.F. is \( y = Ae^{3u} + Be^u \)

For the P.I. try \( y = Ce^{2u} \)

\[
\Rightarrow \quad \frac{dy}{du} = 2Ce^{2u} \quad \text{and} \quad \frac{d^2y}{du^2} = 4Ce^{2u}
\]

\[
\Rightarrow \quad 4Ce^{2u} - 8Ce^{2u} + 3Ce^{2u} = -2e^{2u}
\]

\( C = 2 \)

G.S. is \( y = Ae^{3u} + Be^u + 2e^{2u} \)

But \( x = e^u \) \( \Rightarrow \) G.S. is \( y = Ax^3 + Bx + 2x^2 \)
Maclaurin and Taylor Series

1) Maclaurin series

\[ f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots \]

2) Taylor series

\[ f(x + a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^n}{n!}f^n(a) + \cdots \]

3) Taylor series – as a power series in \((x - a)\)

Replacing \(x\) by \((x - a)\) in 2) we get

\[ f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots \]

4) Solving differential equations using Taylor series

(a) If we are given the value of \(y\) when \(x = 0\), then we use the Maclaurin series with

\[ f(0) = y_0 \quad \text{the value of } y \text{ when } x = 0 \]

\[ f'(0) = \left(\frac{dy}{dx}\right)_0 \quad \text{the value of } \frac{dy}{dx} \text{ when } x = 0 \]

etc. to give

\[ f(x) = y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \cdots + \frac{x^n}{n!}\left(\frac{d^n y}{dx^n}\right)_0 + \cdots \]

(b) If we are given the value of \(y\) when \(x = x_0\), then we use the Taylor power series with

\[ f(a) = y_{x_0} \quad \text{the value of } y \text{ when } x = x_0 \]

\[ f'(a) = \left(\frac{dy}{dx}\right)_{x_0} \quad \text{the value of } \frac{dy}{dx} \text{ when } x = x_0 \]

etc. to give

\[ y = y_{x_0} + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x-x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \frac{(x-x_0)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_{x_0} + \cdots \]

NOTE THAT 4 (a) and 4 (b) are not in the formula book, but can easily be found using the results in 1) and 3).
Standard series

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \] converges for all real \( x \)

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots \] converges for all real \( x \)

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \cdots \] converges for all real \( x \)

\[ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots \] converges for \(-1 < x \leq 1\)

\[ (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots + \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!} x^r + \cdots \] converges for \(-1 < x < 1\)

Example 1: Find the Maclaurin series for \( f(x) = \tan x \), up to and including the term in \( x^3 \)

Solution:

\[ f(x) = \tan x \] \quad \Rightarrow \quad f'(0) = 0

\[ f'(x) = \sec^2 x \] \quad \Rightarrow \quad f''(0) = 1

\[ f''(x) = 2 \sec^2 x \tan x \] \quad \Rightarrow \quad f'''(0) = 0

\[ f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \] \quad \Rightarrow \quad f''''(0) = 2

and \( f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots \)

\[ \Rightarrow \quad \tan x \approx 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2 \quad \text{up to the term in } x^3 \]

\[ \Rightarrow \quad \tan x \approx x + \frac{x^3}{3} \]

Example 2: Using the Maclaurin series for \( e^x \) to find an expansion of \( e^{x^2} \), up to and including the term in \( x^3 \).

Solution:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

\[ \Rightarrow \quad e^{x^2} \approx 1 + (x + x^2) + \frac{(x + x^2)^2}{2!} + \frac{(x + x^2)^3}{3!} \quad \text{up to the term in } x^3 \]

\[ \approx 1 + x + x^2 + \frac{x^2 + 2x^3}{2!} + \frac{x^3}{3!} \quad \text{up to the term in } x^3 \]

\[ \Rightarrow \quad e^{x^2} \approx 1 + x + \frac{3}{2} x^2 + \frac{7}{6} x^3 \quad \text{up to the term in } x^3 \]
Example 3: Find a Taylor series for \( \cot \left( x + \frac{\pi}{4} \right) \), up to and including the term in \( x^2 \).

Solution: \( f(x) = \cot x \) and we are looking for

\[
f \left( x + \frac{\pi}{4} \right) = f \left( \frac{\pi}{4} \right) + x f' \left( \frac{\pi}{4} \right) + \frac{x^2}{2!} f'' \left( \frac{\pi}{4} \right) + \cdots
\]

\[
f(x) = \cot x \quad \Rightarrow \quad f \left( \frac{\pi}{4} \right) = 1
\]

\[
f'(x) = -\csc^2 x \quad \Rightarrow \quad f' \left( \frac{\pi}{4} \right) = -2
\]

\[
f''(x) = 2 \csc^2 x \cot x \quad \Rightarrow \quad f'' \left( \frac{\pi}{4} \right) = 4
\]

\[
\cot \left( x + \frac{\pi}{4} \right) \approx 1 - 2x + \frac{x^2}{2!} \times 4 \quad \text{up to the term in } x^2
\]

\[
\cot \left( x + \frac{\pi}{4} \right) \approx 1 - 2x + 2x^2 \quad \text{up to the term in } x^2
\]

Example 4: Use a Taylor series to solve the differential equation,

\[
y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 + y = 0 \quad \text{equation I}
\]

up to and including the term in \( x^3 \), given that \( y = 1 \) and \( \frac{dy}{dx} = 2 \) when \( x = 0 \).

In this case we shall use

\[
f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots
\]

\[
\Rightarrow \quad y = y_0 + x \left( \frac{dy}{dx} \right)_0 + \frac{x^2}{2!} \left( \frac{d^2y}{dx^2} \right)_0 + \frac{x^3}{3!} \left( \frac{d^3y}{dx^3} \right)_0
\]

We already know that \( y_0 = 1 \) and \( \left( \frac{dy}{dx} \right)_0 = 2 \) values when \( x = 0 \)

\[
\Rightarrow \quad \left( \frac{d^2y}{dx^2} \right)_0 = \left( -\frac{1}{y} \left( \frac{dy}{dx} \right)^2 - 1 \right)_0 = -5 \quad \text{values when } x = 0
\]

Differentiating \( y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 + y = 0 \)

\[
\Rightarrow \quad y \frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0
\]

Substituting \( y_0 = 1 \), \( \left( \frac{dy}{dx} \right)_0 = 2 \) and \( \left( \frac{d^2y}{dx^2} \right)_0 = -5 \) values when \( x = 0 \)

\[
\Rightarrow \quad \left( \frac{d^3y}{dx^3} \right)_0 + 2 \times (-5) + 2 \times 2 \times (-5) + 2 = 0
\]

\[
\Rightarrow \quad \left( \frac{d^3y}{dx^3} \right)_0 = 28
\]

\[
\Rightarrow \quad \text{solution is } y \approx 1 + 2x + \frac{x^2}{2!} \times (-5) + \frac{x^3}{3!} \times 28
\]

\[
\Rightarrow \quad y \approx 1 + 2x - \frac{5}{2} x^2 + \frac{14}{3} x^3
\]
Series expansions of compound functions

Example: Find a polynomial expansion for
\[ \frac{\cos 2x}{1-3x}, \] up to and including the term in \( x^3 \).

Solution: Using the standard series
\[ \cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots \] up to and including the term in \( x^3 \)

and
\[ (1 - 3x)^{-1} = 1 + 3x + \frac{-1 \cdot x \cdot (-3x)^2}{2!} + \frac{-1 \cdot x \cdot (-3x)^3}{3!} \]
\[ = 1 + 3x + 9x^2 + 27x^3 \] up to and including the term in \( x^3 \)

\[ \Rightarrow \frac{\cos 2x}{1-3x} = \left(1 - \frac{(2x)^2}{2!}\right)(1 + 3x + 9x^2 + 27x^3) \]
\[ = 1 + 3x + 9x^2 + 27x^3 - 2x^2 - 6x^3 \] up to and including the term in \( x^3 \)

\[ \Rightarrow \frac{\cos 2x}{1-3x} = 1 + 3x + 7x^2 + 21x^3 \] up to and including the term in \( x^3 \)
Polar Coordinates

The polar coordinates of $P$ are $(r, \theta)$

$r = OP$, the distance from the origin,

and $\theta$ is the angle made anti-clockwise with the initial line.

In the Edexcel syllabus $r$ is always taken as positive

\[
\text{But in most books } r \text{ can be negative, thus } (-4, \frac{\pi}{2}) \text{ is the same point as } (4, \frac{3\pi}{2})
\]

Polar and Cartesian coordinates

From the diagram

\[
r = \sqrt{x^2 + y^2}
\]

and $\tan \theta = \frac{y}{x}$ (use sketch to find $\theta$).

\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

Sketching curves

In practice, if you are asked to sketch a curve, it will probably be best to plot a few points. The important values of $\theta$ are those for which $r = 0$.

The sketches in these notes will show when $r$ is negative by plotting a dotted line; these sections should be ignored as far as Edexcel A-level is concerned.

Some common curves

$r = a + b \cos \theta$

\begin{itemize}
  \item Cardiod $a = b$
  \item Limacon without dimple $a \geq 2b$
  \item Limacon with a dimple $b \leq a < 2b$
\end{itemize}
Limacon with a loop

\[ a < b \]
\[ r \text{ negative in the loop} \]

Circle

\[ r \text{ negative in bottom half} \]

Line

Rose Curves

\[ r = 4 \cos 3\theta \]
\[ 0 \leq \theta \leq \pi \]

\[ r = 4 \cos 3\theta \]
\[ \pi \leq \theta \leq 2\pi \]
Thus the rose curve \( r = a \cos \theta \) always has \( n \) petals, when only the positive values of \( r \) are taken.

Leminiscate of Bernoulli

Spiral \( r = 2\theta \)

Spiral \( r = e^\theta \)

Circle \( r = 10 \cos \theta \)

Notice that in the circle on \( OA \) as diameter, the angle \( P \) is \( 90^\circ \) (angle in a semi-circle) and trigonometry gives us that \( r = 10 \cos \theta \).
Circle \( r = 10 \sin \theta \)

In the same way \( r = 10 \sin \theta \) gives a circle on the \( y \)-axis.

Areas using polar coordinates

Remember: area of a sector is \( \frac{1}{2} r^2 \theta \)

\[
\text{Area of } OQ \approx \delta A \approx \frac{1}{2} r^2 \delta \theta
\]

\[
\Rightarrow \quad \text{Area } OAB \approx \sum \left( \frac{1}{2} r^2 \delta \theta \right)
\]

as \( \delta \theta \to 0 \)

\[
\Rightarrow \quad \text{Area } OAB = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta
\]

Example: Find the area between the curve \( r = 1 + \tan \theta \) and the half lines \( \theta = 0 \) and \( \theta = \frac{\pi}{3} \)

Solution: \[
\text{Area} = \int_{0}^{\pi/3} \frac{1}{2} r^2 \, d\theta
\]
\[
= \int_{0}^{\pi/3} \frac{1}{2} (1 + \tan \theta)^2 \, d\theta
\]
\[
= \int_{0}^{\pi/3} \frac{1}{2} (1 + 2 \tan \theta + \tan^2 \theta) \, d\theta
\]
\[
= \int_{0}^{\pi/3} \frac{1}{2} (2 \tan \theta + \sec^2 \theta) \, d\theta
\]
\[
= \frac{1}{2} [2 \ln(\sec \theta) + \tan \theta]_0^{\pi/3}
\]
\[
= \ln 2 + \frac{\sqrt{3}}{2}
\]
Tangents parallel and perpendicular to the initial line

\[ y = r \sin \theta \quad \text{and} \quad x = r \cos \theta \]

\[ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \]

1) Tangents will be parallel to the initial line \((\theta = 0)\), or horizontal, when \( \frac{dy}{dx} = 0 \)

\[ \Rightarrow \quad \frac{dy}{d\theta} = 0 \]

\[ \Rightarrow \quad \frac{d}{d\theta} (r \sin \theta) = 0 \]

2) Tangents will be perpendicular to the initial line \((\theta = 0)\), or vertical, when \( \frac{dy}{dx} \) is infinite

\[ \Rightarrow \quad \frac{dx}{d\theta} = 0 \]

\[ \Rightarrow \quad \frac{d}{d\theta} (r \cos \theta) = 0 \]

Note that if both \( \frac{dy}{d\theta} = 0 \quad \text{and} \quad \frac{dx}{d\theta} = 0 \), then \( \frac{dy}{dx} \) is not defined, and you should look at a sketch to help (or use l'Hôpital's rule).

**Example:** Find the coordinates of the points on \( r = 1 + \cos \theta \) where the tangents are

(a) parallel to the initial line,

(b) perpendicular to the initial line.

**Solution:** \( r = 1 + \cos \theta \) is shown in the diagram.

(a) Tangents parallel to \( \theta = 0 \) (horizontal)

\[ \Rightarrow \quad \frac{dx}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta} (r \sin \theta) = 0 \]

\[ \Rightarrow \quad \frac{d}{d\theta} ((1 + \cos \theta) \sin \theta) = 0 \quad \Rightarrow \quad \frac{d}{d\theta} (\sin \theta + \sin \theta \cos \theta) = 0 \]

\[ \Rightarrow \quad \cos \theta - \sin^2 \theta + \cos^2 \theta = 0 \quad \Rightarrow \quad 2 \cos^2 \theta + \cos \theta - 1 = 0 \]

\[ \Rightarrow \quad (2 \cos \theta - 1)(\cos \theta + 1) = 0 \quad \Rightarrow \quad \cos \theta = \frac{1}{2} \quad \text{or} \quad -1 \]

\[ \Rightarrow \quad \theta = \pm \frac{\pi}{3} \quad \text{or} \quad \pi \]

(b) Tangents perpendicular to \( \theta = 0 \) (vertical)

\[ \Rightarrow \quad \frac{dx}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta} (r \cos \theta) = 0 \]

\[ \Rightarrow \quad \frac{d}{d\theta} ((1 + \cos \theta) \cos \theta) = 0 \quad \Rightarrow \quad \frac{d}{d\theta} (\cos \theta + \cos^2 \theta) = 0 \]

\[ \Rightarrow \quad -\sin \theta - 2 \cos \theta \sin \theta = 0 \quad \Rightarrow \quad \sin \theta (1 + 2 \cos \theta) = 0 \]

\[ \Rightarrow \quad \cos \theta = -\frac{1}{2} \quad \text{or} \quad \sin \theta = 0 \]

\[ \Rightarrow \quad \theta = \pm \frac{2\pi}{3} \quad \text{or} \quad 0, \pi \]
From the above we can see that

(a) the tangent is parallel to $\theta = 0$

at $B \left( \theta = \frac{\pi}{3} \right)$, and $E \left( \theta = -\frac{\pi}{3} \right)$,

also at $\theta = \pi$, the origin – see below

(b) the tangent is perpendicular to $\theta = 0$

at $A \left( \theta = 0 \right)$, $C \left( \theta = \frac{2\pi}{3} \right)$ and $D \left( \theta = -\frac{2\pi}{3} \right)$

(c) we also have both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ when $\theta = \pi$!!

From the graph it looks as if the tangent is parallel to $\theta = 0$ at the origin, $(\theta = \pi)$, and from l'Hôpital's rule it can be shown that this is true.
complex numbers, 7
applications of De Moivre’s theorem, 8
argument, 7
De Moivre’s theorem, 8
Euler’s relation, 7
loci, 10
loci and geometry, 13
modulus, 7
nth roots, 9
roots of polynomial equations, 10
transformations, 12
differential equations. see second order
 differential equations, see first order
differential equations
first order differential equations, 14
equations, 14
families of curves, 14
integrating factors, 15
separating the variables, 14
using substitutions, 15
inequalities, 3
algebraic solutions, 3
graphical solutions, 4
Maclaurin and Taylor series, 22
expanding compound functions, 25
standard series, 23
worked examples, 23
method of differences, 5
polar coordinates, 26
area, 29
cardiod, 26
circle, 28
lemniscate, 28
polar and cartesian, 26
\[ r = a \cos n\theta, \]
spiral, 28
tangent, 30
second order differential equations, 17
auxiliary equation, 17
complimentary function, 17
general solution, 18
linear with constant coefficients, 17
particular integral, 18
using substitutions, 21