AQA
ASSESSMENT and QUALIFICATIONS ALLIANCE

Not Formulae and Statistical Tables for GCE Mathematics and GCE Statistics

The Not-Formula Book

Further Pure 1

Everything you need to remember that the formula book won't tell you

The Not-Formula Book for FP1

Everything you need to know for Further Pure 1 that *won't* be in the formula book

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use in this module.

Contents

Chapter 1 – Roots of quadratic equations

When the quadratic equation $ax^2 + bx + c = 0$ has roots α and β:

The sum of the roots, $\alpha + \beta = -\frac{b}{a}$

The product of the roots, $\alpha\beta = \frac{c}{a}$

By writing the equation as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ it can be expressed as:

$$x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$

Note: Using the above results and rearranging, the following results can be produced:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

For a known quadratic, another can be formed whose roots are related to the roots of the first by applying the following method:

Step 1: Write down the sum of the roots, $\alpha + \beta$, and the product of the roots, $\alpha\beta$, of the given equation.
Step 2: Find the sum and product of the new roots in terms of $\alpha + \beta$ and $\alpha\beta$.
Step 3: Write down the new equation using $x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$

Eg: Find a quadratic whose roots are the reciprocals of the roots of $2x^2 - 5x - 6 = 0$

Sum and product of original roots:

$$\alpha + \beta = -\frac{b}{a} = \frac{5}{2} \qquad \alpha\beta = \frac{c}{a} = -3$$

Sum and product of new roots:

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{5}{2}}{-3} = -\frac{5}{6} \qquad \frac{1}{\alpha}\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{-3} = -\frac{1}{3}$$

New equation:

$$x^2 \text{—} \frac{5}{6}x + \left(-\frac{1}{3}\right) = 0 \qquad or \qquad x^2 + \frac{5}{6} - \frac{1}{3} = 0$$

Chapter 2 – Complex numbers

There is no real number which can be squared to give a negative, and so the symbol i is used to denote the imaginary number which is equal to the square root of -1. In other words, $i^2 = -1$

Since $i^2 = -1$, it follows that $i^4 = 1$. The powers of i form a periodic cycle:

$$i = i, \qquad i^2 = -1, \qquad i^3 = -i, \qquad i^4 = 1, \qquad i^5 = i, \qquad \textit{and so on}$$

While quadratics of the form $x^2 = -k$ have no real roots, they do have imaginary ones:

$$x = \pm ki$$

A complex number is a number of the form $p + qi$, where p and q are real numbers and $i^2 = -1$. This is essentially a number comprised of both real and imaginary parts.

Note: Either the real or the imaginary part of a complex number may be equal to zero, making the real numbers a subset of the complex numbers. Any purely imaginary number is also a complex number.

The **complex conjugate** of $p + qi$ is $p - qi$, where p and q are real numbers. The conjugate of z is denoted by z^*.

When a quadratic equation with real coefficients has complex roots, these roots are always a pair of complex conjugates.

Note: The easiest way to solve quadratic equations with complex roots is to complete the square.

Eg:

$$x^2 + 6x + 25 = 0 \quad \Longrightarrow \quad (x+3)^2 - 9 + 25 = 0$$

$$\Longrightarrow \quad (x+3)^2 = -16 \quad \Longrightarrow \quad x = -3 \pm \sqrt{-16} = -3 \pm 4i$$

In general, when $z = p + qi$, where p and q are real numbers:

The real part of z is p and the imaginary part of z is q.

Note: This may appear to be self-evident, but it is an important result. If two complex numbers are equal, they must necessarily have both equal real and equal imaginary parts. Solving equations involving complex numbers, then, is equivalent to solving equations involving two-dimensional vectors. The real part of the equation and the imaginary part form their own distinct equations, both of which must be satisfied to solve the original equation.

Chapter 3 – Inequalities

There are important differences between solving equations and solving inequalities:

An inequality will have a *range* of values as its solution.

Whenever you multiply or divide an inequality by a *negative* number you must also *reverse* the inequality sign.

When solving quadratic/cubic/higher order inequalities you must consider **critical values**.

You calculate the value of $f(x)$ in each of the regions on the number line created by the critical values and produce a sign diagram.

Note: The best way of approaching more complex inequalities is to rearrange to produce a function on one side and zero on the other. Then critical values can be found, and a sign diagram produced.

When solving inequalities involving a rational expression, you must ensure that the expression is written in an appropriate form (i.e. factorised/simplified as far as possible and with 0 on the right-hand side).

You can then find the critical values for the numerator and denominator.

There are various methods for solving inequalities involving rational expressions such as:

Method 1: Multiplying throughout by the square of the denominator.
Method 2: Combining all the fractions into one single term on one side of the inequality.

Note: While it may seem simplest to multiply by the denominator, due to the nature of inequalities where negatives are involved, the best way to accomplish the same thing is to multiply by its square, thus ensuring it is always positive regardless of the value of x.

Chapter 4 – Matrices

A **matrix** is a rectangular array of numbers. Each entry in the matrix is called an **element**.

A matrix with m rows and n columns is an $m \times n$ matrix. This is called the **order** of the matrix.

We can add or subtract matrices provided they have the **same order**.

To add or subtract matrices, all that is required is that we add or subtract **corresponding elements** from each matrix.

Eg:

$$\begin{bmatrix} 3 & 2 & 5 \\ -1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 1 \\ 0 & -4 & 10 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 4 \\ -1 & 4 & -8 \end{bmatrix}$$

To multiply a matrix by a constant, simply multiply each element of the matrix by the constant.

Note: This is equivalent to finding the scalar multiple of a vector – in fact, that process is just a specific example of multiplying a matrix by a constant.

Eg:

$$5\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 5 \end{bmatrix}$$

We can multiply two matrices **A** and **B** only if the number of columns of **A** equals the number of rows of **B**.

Note: This does not mean the matrices have to be of equal order, but due to the method used for multiplying matrices, they must fulfil this requirement.

If **A** is a matrix of order $a \times b$ and **B** is a matrix of order $c \times d$, then the matrix **AB** exists if and only if $b = c$.

The product **AB** will have order $a \times d$.

Note: **AB** means **A** multiplied by **B**, in that order. $\boldsymbol{A^2} = \boldsymbol{AA} \quad and \quad \boldsymbol{A^3} = \boldsymbol{AAA} = (\boldsymbol{AA})\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{AA})$

To multiply two matrices, first generate the correct size for the result matrix as described above. To find the value of the each element in this matrix, calculate the sum of the products of the elements in the corresponding row of the first matrix and column of the second.

That is, to find the element in the third row and second column, for instance, you would multiply each element in the first matrix's third row by the corresponding element in the second matrix's second column.

Eg:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \quad \boldsymbol{AB} = \begin{bmatrix} 3 \times -1 + 4 \times 3 & 3 \times 0 + 4 \times 2 \\ 5 \times -1 + 6 \times 3 & 5 \times 0 + 6 \times 2 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 13 & 12 \end{bmatrix}$$

In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$. Matrix multiplication is not, in general, commutative.

Note: The terms *pre-multiplied* and *post-multiplied* can be used to indicate which way round matrices are to be multiplied. For instance, in **AB**, **A** is *post-multiplied* by **B**, or, equivalently, **B** is *pre-multiplied* by **A.**

A matrix which has the same number of rows and columns is called a **square matrix**.

The matrix $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix **A** by **I** you get **A** as the answer.

This means that for any 2×2 matrix **A**: $\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$

More generally, an identity matrix can be formed for any size of square matrix.

Eg:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \textit{ is the } 3 \times 3 \textit{ identity matrix}$$

The square matrix where every element is zero is called the **zero matrix**. For instance, the 2×2 zero matrix is:

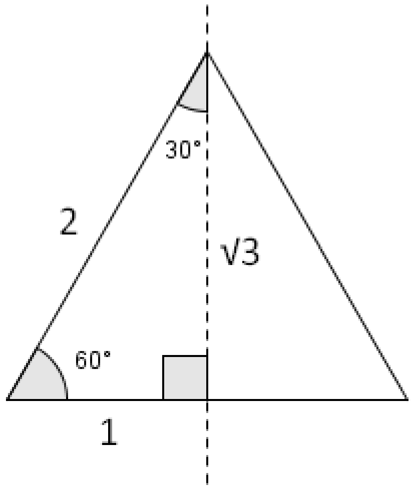
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

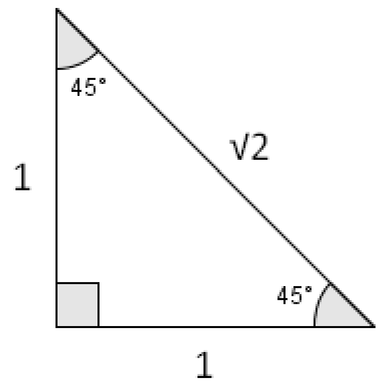
Adding the zero matrix to any other matrix leaves it unchanged, and multiplying it by any other matrix gives the zero matrix.

Note: The identity matrix is the equivalent of the number 1 – anything multiplied by it remains unchanged. The zero matrix is the equivalent of the number 0 – anything added to it remains unchanged, and anything multiplied by it becomes the zero matrix.

Chapter 5 - Trigonometry

By applying right-angle trigonometry to a bisected equilateral triangle of side length 2 or an isosceles right-angled triangle (equal sides length 1), the following exact results can be obtained – these are preferable to approximations given by calculators and should be used where possible:





θ *in degrees*	θ *in radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	∞
180	π	0	-1	0
360	2π	0	1	0

To find the **general solution of $\sin x = k$** or **$\cos x = k$**, where $-1 \leq k \leq 1$, you find two solutions lying in the interval $-180° \leq x \leq 180°$, and then, since sine and cosine are periodic with period $360°$, you add $360n°$, where n is an integer, to each of the two solutions.

For x in **radians**, you find the general solution of $\sin x = k$ and $\cos x = k$, where $-1 \leq k \leq 1$, by finding two solutions lying in the interval $-\pi \leq x \leq \pi$ and then adding $2n\pi$, where n is an integer, to each of the two solutions.

The **general solution of $\cos x = \cos \alpha$** is $x = 2n\pi \pm \alpha$ (or $x = (360n \pm \alpha)°$, if x is in degrees), where n is an integer.

The **general solution of $\sin x = \sin \alpha$** is $x = 2n\pi + \alpha, 2n\pi + \pi - \alpha$ (or $x = (360n + \alpha)°, (360n + 180 - \alpha)°$, if x is in degrees), where n is an integer.

To find the **general solution of $\tan x = k$**, where k is a constant, you find the solution lying in the interval $-90° \leq 90°$ and then, since tan is periodic with period $180°$, you add $180n°$, where n is an integer, to the solution.

For x in **radians** you find the general solution of $\tan x = k$ by finding the solution lying in the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ then adding $n\pi$, where n is an integer, to the solution.

The **general solution of $\tan x = \tan \alpha$** is $x = n\pi + \alpha$ (or $x = (180n + \alpha)°$, if x is in degrees), where n is an integer.

Chapter 6 – Matrix transformations

A **linear transformation** that changes/transforms point $P(x, y)$ into point $P'(x', y')$ can be written as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{matrix\ form}$$

or:

$$x' = ax + by \qquad y' = cx + dy \quad \boldsymbol{algebraic\ form}$$

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called a transformation matrix.

Note: multiplying the matrix by the original vector will convert from the matrix form to the algebraic form, which may be easier to interpret depending on the situation.

A **one-way stretch** in the $\boldsymbol{x}$**-direction** of scale factor k is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A **one-way stretch** in the $\boldsymbol{y}$**-direction** of scale factor k is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A **two-way stretch** of scale factor a in the x-direction and scale factor b in the y-direction is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: There is a similarity in the stretching transformation matrix to the identity matrix. The 0 elements ensure that the new x values are not affected by the old y values, and vice versa. The elements on the leading diagonal (top left to bottom right) determine the relationship between the new x and y values and the old ones.

An enlargement of scale factor k with centre the origin is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: This is a specific example of a two-way stretch (where the x- and y-direction scale factors are equal).

Note: **The following rotation and reflection matrices *are* included in your formula book.** (Included here for completeness only)

A **rotation** through angle θ anticlockwise about the origin is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: This transformation matrix is derived by considering the effect of the transformation on the points (1,0) and (0,1). Trigonometry gives the desired result.

A **reflection** in the line $y = x\tan\theta$ (where θ is the angle the line makes with the positive x-axis) is given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: Again, this transformation is derived from considering the points (1,0) and (0,1). This matrix has certain similarities with the rotation matrix. This is because the two transformations do a similar thing mathematically. Any transformation produced through a rotation, for instance, can also be produced by a combination of two reflections.

If the 2×2 matrices, **A** and **B**, both represent transformations, then the product matrix **BA** also represents a transformation equivalent to applying **A** followed by **B**.

This type of transformation is known as a **composite transformation**.

It is important to remember to read composite transformations backwards, i.e., **ABC** means a transformation produced by applying first **C**, then **B** and finally **A**.

Chapter 7 – Linear laws

Since experimental data is not exact, due to measuring errors, points plotted are unlikely to all lie exactly in line so a line of best fit is drawn. Remember that a line of best fit will pass through the mean (average of x values , average of y values).

Note: throughout this section you will need to recall the general equation for a straight line:

$$y = mx + c \quad where \quad c = y - intercept \quad and \quad m = gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

If the variables used are not x and y, the method to find the equation of the line of best fit is exactly the same. In the general equation $y = mx + c$ you just replace y by the variable on the vertical axis and replace x by the variable on the horizontal axis.

If the graph does not show the y-intercept, you can find the value of the gradient m as usual and then find the value of c by substituting the coordinates of a point on the line into the equation $y = mx + c$. Alternatively, you can use the coordinates of two points on the line to form and solve a pair of simultaneous equations in m and c.

Linear laws allow us to compare various non-linear relationships between variables. All that is required is a rewriting of the original relationship so that is of the form:

$$Y \quad = \quad mX \quad + \quad c$$

$$Variable \ = \ Variable \ + \ Constant$$

Where X and Y are variables which do not need to be simply x and y. Examples are shown below:

To test a belief that the relation between x and y is of the form $\boldsymbol{y = ax^2 + b}$, you need to plot y against x^2. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^2 + b$. The gradient of the line gives an estimate for a and the intercept on the vertical axis ($X = 0$) gives an estimate for b.

To test a belief that the relation between x and y is of the form $\boldsymbol{\frac{1}{x} + \frac{1}{y} = a}$, you need to plot $\frac{1}{y}$ against $\frac{1}{x}$. If the points are roughly in a straight line with gradient -1, you can deduce that the relation between x and y is of the form $\frac{1}{x} + \frac{1}{y} = a$. The intercept on the vertical axis ($X = 0$) gives an estimate for a.

To test a belief that the relation between x and y is of the form $\boldsymbol{y = ax^2 + bx}$, you can plot $\frac{y}{x}$ against x. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^2 + bx$. The gradient of the line gives an estimate for a and the intercept on the vertical axis ($X = 0$) gives an estimate for b.

When dealing with equations of the form $\boldsymbol{y = ax^n}$, it is necessary to use logarithms to reduce to a linear form:

$$y = ax^n \quad \Longrightarrow \quad \log y = \log a + n \log x$$

To test a belief that the relation between x and y is of the form $y = ax^n$, you need to plot $\log y$ against $\log x$. If the points are roughly in a straight line, you can deduce that the relation between x and y is of the form $y = ax^n$. The gradient of the line gives an estimate for n and the intercept on the vertical axis ($X = 0$) gives the value of $\log a$ from which the estimate for a can be found.

When dealing with equations of the form $\boldsymbol{y = ab^x}$, it is necessary to use logarithms to reduce to a linear form:

$$y = ab^x \quad \Longrightarrow \quad \log y = (\log b)x + \log a$$

To represent the relation $y = ab^x$ in a linear form you need to plot $\log y$ against x. If a straight line is obtained from the given data the relation is true. The gradient of the line is our estimate for $\log b$ and the intercept on the vertical axis ($X = 0$) gives an estimate for $\log a$. Knowing these two values, estimates for a and b can be found.

Chapter 8 – Calculus

The gradient of the chord PQ is given by:

$$\frac{y_Q - y_P}{x_Q - x_P}$$

When the x-coordinate of P is a, and the x-coordinate of Q is $a + h$, the gradient of the chord PQ can be simplified to an expression involving h. The gradient of the curve at the point P is obtained by letting h tend to zero.

Eg: Find the gradient of the function $y = 3x^2 + 2x$ at $x = 4$

$$Let\ the\ point\ be\ P \quad \Longrightarrow \quad x_P = 4 \quad and \quad y_P = 3(4)^2 + 2(4) = 56$$

$$Define\ the\ point\ Q\ to\ be\ at\ x = 4 + h$$
$$\Longrightarrow \quad x_Q = 4 + h \quad and \quad y_Q = 3(4 + h)^2 + 2(4 + h) = 56 + 26h + 3h^2$$

$$Gradient = \frac{(56 + 26h + 3h^2) - (56)}{(4 + h) - (4)} = \frac{26h + 3h^2}{h} = 26 + 3h$$

$$Gradient\ of\ curve\ at\ P = \lim_{h \to 0}(26 + 3h) = 26$$

When $f(x)$ is defined for $x \geq a$, where the limit exists, we define the improper integral:

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} f(x)\,dx$$

When $f(x)$ is defined for $x \leq b$, provided the integral exists, we define the improper integral:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to \infty} \int_a^b f(x)\,dx$$

When $f(x)$ is defined for $p < x < q$, but $f(x)$ is not defined when $x = p$, then the improper integral, provided the limit exists, is given by:

$$\int_p^q f(x)\,dx = \lim_{a \to p+} \int_a^q f(x)\,dx$$

Chapter 9 – Series

Sum of natural numbers – In formula book, included for completeness:

$$1+2+3+4+\cdots+n=\sum_{r=1}^{n} r=\frac{n}{2}(n+1)$$

Note: The summation $\sum_{r=a}^{b} f(r)$ means the sum of all values of $f(r)$ from $f(a)$ to $f(b)$:

$$\sum_{r=a}^{b} f(r)=f(a)+f(a+1)+\cdots+f(b)$$

$$\sum_{r=1}^{n} 1=1+1+\cdots+1=n \quad (\textit{there are } n \textit{ terms added together})$$

Note: The proof of the following two formulae is not required for this module, although you will be expected to apply the results:

Sum of squares – In formula book, included for completeness:

$$1^2+2^2+3^2+4^2+\cdots+n^2=\sum_{r=1}^{n} r^2=\frac{n}{6}(n+1)(2n+1)$$

Sum of cubes – In formula book, included for completeness:

$$1^3+2^3+3^3+4^3+\cdots+n^3=\sum_{r=1}^{n} r^3=\frac{n^2}{4}(n+1)^2$$

A series can often be split into separate, simpler, summations:

$$\sum[Af(r)+Bg(r)]=A\sum f(r)+B\sum g(r)$$

Eg: Show that $\sum_{r=1}^{4} 3r-r^2=0$

$$\sum_{r=1}^{4} 3r-r^2=3\sum_{r=1}^{4} r-\sum_{r=1}^{4} r^2=3\left(\frac{4}{2}(4+1)\right)-\left(\frac{4}{6}(4+1)(2\times 4+1)\right)=30-30=0$$

Note: A partial summation (starting from a number other than 1) can be found by splitting into two.

Eg: Find the value of $\sum_{r=5}^{10}(2r^2+r)$

$$\sum_{r=5}^{10}(2r^2+r)=\sum_{r=1}^{10}(2r^2+r)-\sum_{r=1}^{4}(2r^2+r)$$

Chapter 10 – Numerical methods

If the graph of $y = f(x)$ is continuous over the interval $a \leq x \leq b$, and $f(a)$ and $f(b)$ have different signs, then a root of the equation $f(x) = 0$ must lie in the interval $a < x < b$.

Note: This result is self-evident; if the line of a graph doesn't jump, and has values both above and below the x-axis, it must at some point cross (at least once, but could be any odd number of times).

The bisection method

When a root of $f(x) = 0$ is known to lie between $x = a$ and $x = b$ as above, the bisection method requires you to next find the value of $f(\frac{a+b}{2})$
This point, combined with whichever of the first points has the opposite sign, provide the new pair of values between which the root must lie.
The procedure is repeated until you have an interval of the desired width containing the root.

Note: This method halves the size of the interval with every repetition. This is very efficient, since the interval will be a millionth the size after only 20 repetitions.

Linear interpolation

When a root of the equation $f(x) = 0$ is known to lie between $x = a$ and $x = b$, linear interpolation involves replacing the curve by a straight line and gives an approximation to the root as:

$$\frac{af(b) + bf(a)}{f(b) - f(a)}$$

Note: This method is often more effective than the bisection method, since it takes into account the average gradient of the curve between the two points, thus generally providing closer approximations with fewer repetitions.

Newton-Raphson iteration – Given in the formula book, included for completion only.

The Newton-Raphson iterative formula for solving $f(x) = 0$ is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note: This method improves on the previous two in that it uses calculus to predict the location of the root. Essentially it involves finding the gradient of the curve at the starting point, and projecting a tangent from this point to the x-axis. The crossing point of this tangent line is the next approximation. Using this better approximation, a further approximation can be found, and so on.

Note: You also need to be able to apply Euler's step-by-step method, but this is in the formula book.

Chapter 11 – Asymptotes and rational functions

An asymptote is a line that a curve approaches for large values of $|x|$ or $|y|$. It is usually represented by a broken or dotted line.

The line $x = a$ is a vertical asymptote of the curve $y = \frac{f(x)}{g(x)}$ if $g(a) = 0$.

To find a horizontal asymptote of $y = \frac{ax+b}{cx+d}$, rewrite the equation as:

$$y = \frac{a + \frac{b}{x}}{c + \frac{d}{x}}$$

As $|x| \to \infty$, $\frac{1}{x} \to 0$, therefore $y \to \frac{a}{c}$. The horizontal asymptote has equation $y = \frac{a}{c}$.

In order to solve inequalities such as $\frac{ax+b}{cx+d} < k$ or $\frac{ax+b}{cx+d} > k$:

1 Sketch the graph of $y = \frac{ax+b}{cx+d}$ and the line $y = k$.
2 Solve the equation $\frac{ax+b}{cx+d} = k$.
3 Use the graph to find the possible values of x for which the graph lies below or above the line.

Chapter 12 – Further rational functions and maximum and minimum points

It is always useful to draw any asymptotes as the first stage in sketching the graph of a rational function.

When a curve has a vertical asymptote at $x = a$, it is useful to check the values of y when x is a little smaller than a and when x is a little larger than a.

By considering the behaviour very close to the asymptotes, it is often possible to deduce the main shape of the graph.

In order to find the set of values of y for which a curve of the form $y = \frac{ax^2+bx+c}{dx^2+ex+f}$ exists:

1 Consider where $y = k$ cuts the curve by writing $k = \frac{ax^2+bx+c}{dx^2+ex+f}$.
2 Multiply out to obtain a quadratic of the form $Ax^2 + Bx + C = 0$, where A, B and C will involve k.
3 Use the condition for real roots $B^2 - 4AC \geq 0$ to obtain a quadratic inequality involving k.
4 Convert the solution involving k to a condition involving y.

If a curve of the form $y = \frac{ax^2+bx+c}{dx^2+ex+f}$ is shown to exist only for $P \leq y \leq Q$, then it means that the curve must have a minimum point when $y = P$ and a maximum point when $y = Q$.

Substitute $y = P$ in order to find the x-coordinate of the minimum point. The resulting quadratic in x will always have a repeated root.

Repeat by substituting $y = Q$ to find the x-coordinate of the maximum point.

If a curve of the form $y = \frac{ax^2+bx+c}{dx^2+ex+f}$ is shown to exist only for $y \leq M$ or $y \geq N$, then it means that the curve must have a maximum point when $y = M$ and a minimum point when $y = N$.

Substitute $y = M$ in order to find the x-coordinate of the maximum point. The resulting quadratic in x will always have a repeated root.

Repeat by substituting $y = N$ to find the x-coordinate of the minimum point.

Chapter 13 – Parabolas, ellipses and hyperbolas

A **reflection** in the line $y = x$ maps (x, y) onto (x', y'), where $x' = y$ and $y' = x$.
The equation of the new curve after a reflection in the line $y = x$ is obtained by interchanging x and y in the original equation.

A **parabola** with its vertex at the origin and its axis along the x-axis will have an equation of the form $y^2 = kx$, where k is a constant.

A **stretch** of scale factor c in the x-direction and scale factor d in the y-direction, maps (x, y) onto (x', y'), where $x' = cx$ and $y' = dy$.
The equation of the new curve is obtained by replacing x by $\left(\frac{x}{c}\right)$ and y by $\left(\frac{y}{d}\right)$ in the original equation.

The general equation of an **ellipse** with its centre at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
It cuts the x-axis when $x = \pm a$ and cuts the y-axis when $y = \pm b$.

An equation of the form $xy = c^2$ represents a **hyperbola** with the coordinate axes as asymptotes.
The asymptotes are perpendicular and it is often referred to as a **rectangular hyperbola**.
It can be rotated through 45° to give an equation of the form $x^2 - y^2 = k$.

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a **hyperbola** with centre at the origin cutting the x-axis when $x = \pm a$. It does not intersect the y-axis and its asymptotes have equations $y = \pm\frac{b}{a}x$. When $b = a$ the hyperbola is said to be a **rectangular hyperbola**.

A **translation** with vector $\begin{bmatrix} c \\ d \end{bmatrix}$ maps (x, y) onto (x', y'), where $x' = x + c$ and $y' = y + d$.
The equation of the new curve is obtained by replacing x by $(x - c)$ and y by $(y - d)$ in the original equation.

To consider how a straight line intersects a parabola, ellipse or hyperbola:
1 Form a quadratic equation (usually in terms of x) of the form $ax^2 + bx + c = 0$.
2 When $b^2 - 4ac < 0$ there are no points of intersection.
3 When $b^2 - 4ac > 0$ there are two distinct points of intersection.
4 When $b^2 - 4ac = 0$ there is a single point of intersection. The line is tangent to the curve at that point.

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