Mechanics 4

Revision Notes

March 2014

Mechanics 4

1 Relative motion

Relative displacement

The displacement of A relative to B, is the position of A taking B as the origin, i.e. $\overrightarrow{BA}$, the vector from B to A

so ${}_A\mathbf{r}_B = \mathbf{r}_A - \mathbf{r}_B$.

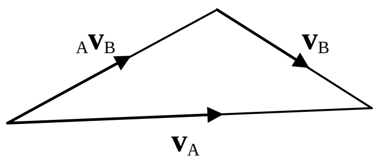
Relative velocity

Differentiating ${}_A\mathbf{r}_B = \mathbf{r}_A - \mathbf{r}_B$

we get ${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B$

We can draw a vector triangle –

sometimes it might be easier to write ${}_A\mathbf{v}_B + \mathbf{v}_B = \mathbf{v}_A$



Also there are times when it is better to write the vectors as column vectors.

The velocity of A **relative to B is the velocity of A assuming that B is stationary**.

Example: A car, C, is travelling north at a speed of 40 km h^{-1}. A truck, T, is travelling south east at a speed of $20\sqrt{2}$ km h^{-1}.

Find the velocity of the truck relative to the car.

Solution: In this case choose vectors $\underline{\boldsymbol{i}}$, east and $\underline{\boldsymbol{j}}$, north.

$$\mathbf{v}_C = \begin{pmatrix} 0 \\ 40 \end{pmatrix}, \quad \mathbf{v}_T = \begin{pmatrix} 20 \\ -20 \end{pmatrix}$$

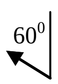
$$\Rightarrow \quad {}_T\mathbf{v}_C = \mathbf{v}_T - \mathbf{v}_C = \begin{pmatrix} 20 \\ -20 \end{pmatrix} - \begin{pmatrix} 0 \\ 40 \end{pmatrix} = \begin{pmatrix} 20 \\ -60 \end{pmatrix}$$

$$\Rightarrow \quad {}_T\mathbf{v}_C = \sqrt{20^2 + 60^2} = 63{\cdot}2 \text{ km h}^{-1}\text{, on a bearing of } 90 + 71{\cdot}6 = 161{\cdot}6^{\circ}.$$

Example: A boat can travel at 4 km h^{-1} in still water. A current flows at 3 km h^{-1} in the direction N 60^0 W.

In what direction should the boat steer so that it travels due north? At what speed will it then be travelling?

Solution: $\mathbf{v}_B$? ↑

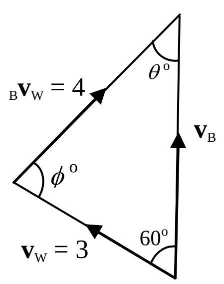
$\mathbf{v}_W$ 3 

${}_B\mathbf{v}_W$ 4 ?

Draw a vector triangle,

${}_B\mathbf{v}_W + \mathbf{v}_W = \mathbf{v}_B$

draw $\mathbf{v}_W$ *first, as we know all about* $\mathbf{v}_W$



Sine Rule $\frac{\sin\theta}{3} = \frac{\sin 60}{4}$

$\Rightarrow \sin\theta = \frac{3\sqrt{3}}{8} \quad \Rightarrow \theta = 40{\cdot}505\ldots^{o}$

$\Rightarrow \phi = 180 - 60 - \theta = 79{\cdot}494\ldots$

and $\frac{\mathbf{v}_B}{\sin 79{\cdot}494\ldots} = \frac{4}{\sin 60} \quad \Rightarrow \mathbf{v}_B = 4{\cdot}541\ldots$

The boat should steer on a bearing of $040{\cdot}5^{o}$, and will travel at $4{\cdot}54$ km h^{-1}.

Collisions

Example: A cyclist, *C*, travelling at 6 m s^{-1} sights a walker, *W*, 500 m due east. The walker is travelling at 2 m s^{-1} on a bearing of 310^{o}. There are no obstacles and both the cyclist and the walker can travel anywhere.

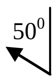
What course should the cyclist set in order to meet the walker, and how long will it take for them to meet?

Solution:

Imagine that W is ***fixed****, then C will travel directly towards W, in this case due east.*

So the direction of ${}_C\mathbf{v}_W$ *will be due* ***east****.*

$\mathbf{v}_C$ 6 ?

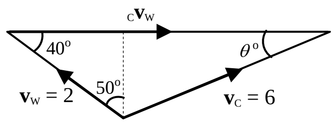
$\mathbf{v}_W$ 2 

${}_C\mathbf{v}_W$? 

Draw a vector triangle,

${}_C\mathbf{v}_W + \mathbf{v}_W = \mathbf{v}_B$

draw $\mathbf{v}_W$ *first, as we know all about* $\mathbf{v}_W$



Sine Rule $\frac{\sin\theta}{2} = \frac{\sin 40}{6}$

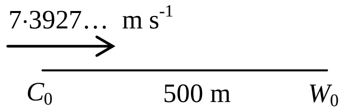
$\Rightarrow \sin\theta = 0{\cdot}2142\ldots \quad \Rightarrow \theta = 12{\cdot}372\ldots^{o}$ so *C* travels on bearing of $90 - 12{\cdot}4 = 77{\cdot}6$.

From the triangle $\left|{}_C\mathbf{v}_W\right| = 2\cos 40 + 6\cos\theta = 7{\cdot}3927\ldots$

Considering the walker as fixed, the cyclist has to travel at $7{\cdot}3927\ldots$ m s^{-1} for a distance of 500 m.

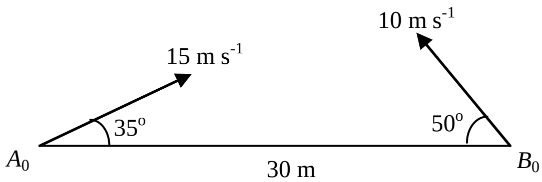
$\Rightarrow$ time to meet $= 500 \div 7.3927\ldots = 67.63\ldots$

Cyclist travels on a bearing of 078^{o} and they meet after 68 seconds.



Closest distance

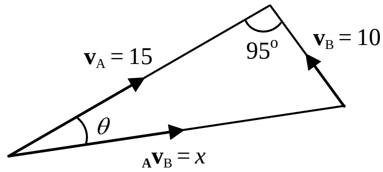
Example: Two ice-skaters, Alice and Bob, start 30 m apart and travel on converging courses. Alice travels at 15 m s^{-1} at an angle of 35° to the initial line and Bob travels at 10 m s^{-1} at an angle of 50° to the initial line, as shown in the diagram.



Find the closest distance between the two skaters, and the time for A to reach that position.

Solution: We consider B as fixed and A moving with the velocity ${}_A\mathbf{v}_B$.

We draw a vector triangle, starting with $\mathbf{v}_A - \mathbf{v}_B$, noting that the angle between $\mathbf{v}_A$ and $\mathbf{v}_B$ is $180 - 35 - 50 = 95^{\circ}$.



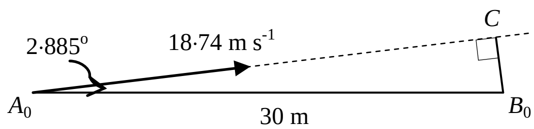
Cosine rule $\quad x^2 = 225 + 100 - 300 \cos 95$

$\Rightarrow \quad x = 18{\cdot}7389\ldots$

Sine rule $\quad \dfrac{\sin\theta}{10} = \dfrac{\sin 95}{x}$

$\Rightarrow \quad \sin\theta = 0{\cdot}5316\ldots \qquad \Rightarrow \qquad \theta = 32{\cdot}114\ldots$

We now think of A moving with speed $18{\cdot}7389\ldots$ $^{\text{m s-1}}$



at an angle of $35 - 32.114\ldots = 2{\cdot}885\ldots^{\circ}$ to the initial line, A_0B_0, with ***B* fixed**.

The closest distance is $B_0C = 30 \sin 2{\cdot}885\ldots = 1.51$ m to 3 S.F.

and A 'moves' with speed $18{\cdot}7389\ldots$ through a distance $A_0C = 30 \cos 2.885\ldots$

$\Rightarrow \qquad A$ takes $\dfrac{30\cos 2{\cdot}885\ldots}{18{\cdot}7389\ldots} = 1{\cdot}60$ seconds to 3 S.F.

B, of course, takes the same time to reach the position where they are closest (just in case you did not realise!).

Best course

Example: A man, who can swim at 2 m s^{-1} in still water, wishes to cross a river which is flowing at 3 m s^{-1}. The river is 40 metres wide, and he wants to drift downstream as little as possible before landing on the other bank.

(*a*) What course should he take?

(*b*) How far downstream does he drift?

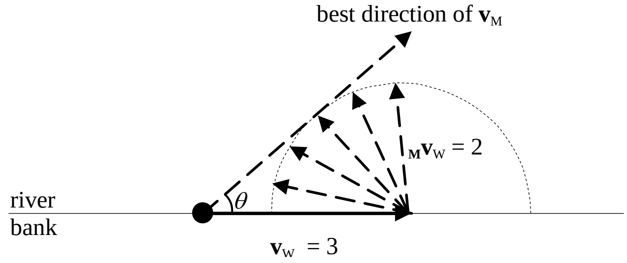
(*c*) How long does it take for him to cross the river.

Solution: The velocity of the man, $\mathbf{v}_M$, must be directed at as big an angle to the downstream bank as possible.

(*a*) $_{M}\mathbf{v}_{W} + \mathbf{v}_{W} = \mathbf{v}_{M}$

First draw $\mathbf{v}_W$ of length 3.

$_{M}\mathbf{v}_{W}$ is of length 2, but can vary in direction.

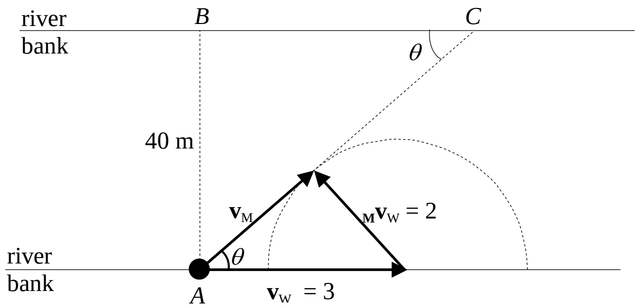


We can choose any direction for $_{M}\mathbf{v}_{W}$, to give $\mathbf{v}_M = {}_{M}\mathbf{v}_{W} + \mathbf{v}_W$

The best direction of $\mathbf{v}_M$ is tangential to the circle of radius 2

$\Rightarrow$ direction of $\mathbf{v}_M$ is at an angle $\theta = \sin^{-1}\left(\frac{2}{3}\right) = 41{\cdot}8\ldots^{\circ}$, with the bank downstream

and the direction of $_{M}\mathbf{v}_{W}$ is at an angle $90 - \theta = 48{\cdot}18\ldots^{\circ}$ with the bank upstream.



The man should swim upstream at an angle of 48° to the bank.

(*b*) Distance downstream is *BC*

$= \frac{40}{\tan\theta} = 20\sqrt{5}$

$= 44{\cdot}7$ m to 3 S.F.

(*c*) $|\mathbf{v}_M| = \sqrt{3^2 - 2^2} = \sqrt{5}$

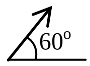
$AC = \frac{40}{\sin\theta} = 60$

$\Rightarrow$ time taken $= \frac{60}{\sqrt{5}} = 12\sqrt{5} = 26{\cdot}8$ s to 3 S.F.

Change in apparent direction of wind or current

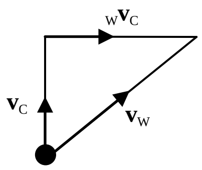
Example: A cyclist travelling due north at 10 m s^{-1}, feels that the wind is coming **from** the west. When travelling in the opposite direction at the same speed the wind appears to be coming **from** a bearing of 210°. What is the true velocity of the wind?

Solution:

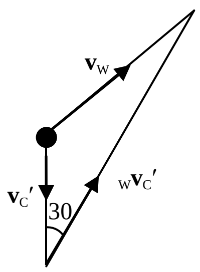
	Case 1			Case 2	
$\mathbf{v}_C$	10	$\uparrow$	$\mathbf{v}_C'$	10	$\downarrow$
$\mathbf{v}_W$	?	?	$\mathbf{v}_W$	?	?
${}_W\mathbf{v}_C$	?	$\rightarrow$	${}_W\mathbf{v}_C'$	?	

Note that $\mathbf{v}_W$ is the same in both cases

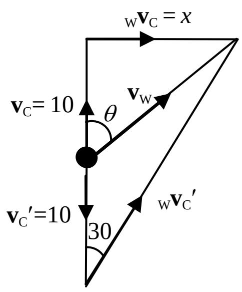
$\mathbf{v}_C + {}_W\mathbf{v}_C = \mathbf{v}_W$



$\mathbf{v}_C' + {}_W\mathbf{v}_C' = \mathbf{v}_W$



Combining the two diagrams



$$x = 20 \tan 30 = \frac{20}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta = \frac{x}{10} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = 49{\cdot}1066\ldots$$

and $|\mathbf{v}_W| = \sqrt{\left(\frac{20}{\sqrt{3}}\right)^2 + 10^2} = \sqrt{\frac{700}{3}} = 15{\cdot}275\ldots$

The true velocity of the wind is 15·3 m s^{-1} blowing **in** the direction 049°.

Note. In this sort of question look for 'nice symmetry', right angled triangles, isosceles triangles or equilateral triangles etc.

2 Elastic collisions in two dimensions

Sphere and flat surface.

- Impulse is along the line perpendicular to the surface through the centre of the sphere.
- Momentum $\perp$ to the surface obeys $I = mv - mu$
- Newton's Experimental Law, NEL, applies to velocity components $\perp$ to the surface.
- Velocity components parallel to the surface remain unchanged.
- **A good diagram showing before, after (and during) is essential!**

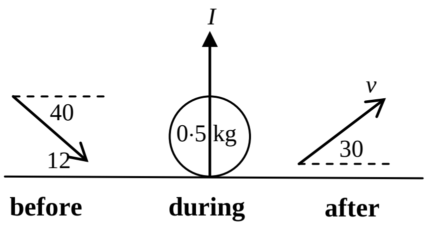
Example: A ball of mass 0·5 kg strikes a smooth, horizontal floor with a velocity of 12 m s^{-1} at 40^{o} to the horizontal, and rebounds at 30^{o} to the horizontal. Calculate

(*a*) the speed of the ball immediately after impact,

(*b*) the coefficient of restitution, *e*

(*c*) the impulse of the floor on the ball.

Solution:



(*a*) Velocities parallel to floor not changed

$\Rightarrow$ $12 \cos 40 = v \cos 30$

$\Rightarrow$ $v = 10{\cdot}6146\ldots = 10{\cdot}6$ m s^{-1} to 3 S.F.

(*b*) Motion perpendicular to the floor

NEL $e = \dfrac{v \sin 30}{12 \sin 40} = 0{\cdot}688059\ldots = 0{\cdot}688$ to 3 S.F.

(*c*) $\uparrow^{+}$ $I = mv - mu = 0{\cdot}5 \times v \sin 30 - 0{\cdot}5 \times (-12 \sin 40) = 6{\cdot}51038\ldots$

$\Rightarrow$ impulse from the floor is 6·51 Ns to 3 S.F.

Colliding spheres in two dimensions

- Impulse is along the line of centres of the spheres.
- Momentum is conserved in all directions – particularly along the line of centres.
- Newton's Experimental Law, NEL, applies to velocity components along the line of centres.
- Velocity components perpendicular to the line of centres remain unchanged.
- **A good diagram showing before, after (and during) is essential!**

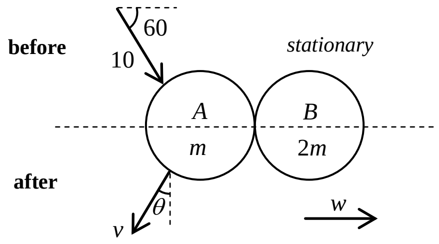
Oblique collisions

In an oblique collision the velocity of one or both spheres is at an angle to the line of centres.

It is often helpful to draw a diagram with the line of centres across the page.

Example: A smooth sphere, A of mass m kg, is moving at a speed of 10 m s^{-1} when it strikes a stationary sphere, B of mass $2m$ kg. The spheres are of equal size, and A is moving at an angle of 60° to the line of centres before impact. The coefficient of restitution is 0·6. Find the velocities of each sphere immediately after the collision.

Solution:



Motion perpendicular to AB is unchanged

$\Rightarrow \quad v\cos\theta = 10\sin 60 = 5\sqrt{3}$

and B moves along the line AB.

For motion parallel to AB

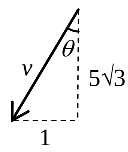
CLM $\quad m \times 10\cos 60 = 2mw - m \times (-v\sin\theta)$

$\Rightarrow \quad 2w + v\sin\theta = 5 \qquad$ **I**

NEL $\quad e = 0{\cdot}6 = \dfrac{w + v\sin\theta}{10\cos 60}$

$\Rightarrow \quad w + v\sin\theta = 3 \qquad$ **II**

I – II $\quad w = 2 \quad \Rightarrow \quad v\sin\theta = 1$



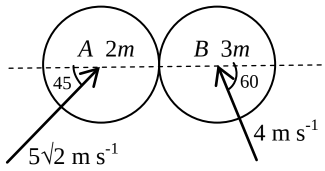
But $v\cos\theta = 5\sqrt{3}, \Rightarrow v = \sqrt{75 + 1^2} = \sqrt{76} = 8{\cdot}7177\ldots$

and $\quad \tan\theta = \frac{1}{5\sqrt{3}}, \Rightarrow \theta = 6.58677\ldots\ldots$

A is moving at 8·72 m s^{-1} at an angle of $96{\cdot}6^{\circ}$ to the line of centres, AB,

and B is moving at 2 m s^{-1} along the line of centres.

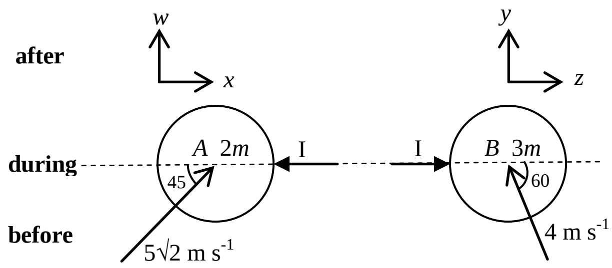
Example: A smooth sphere A of mass $2m$ kg collides with a smooth sphere B of mass $3m$ kg and of equal radius. Just before the collision A is moving with a speed of $5\sqrt{2}$ m s^{-1} in a direction of 45^o to the line of centres, and B is moving with a speed of 4 m s^{-1} at 60^o to the line of centres, as shown in the diagram. The coefficient of restitution is $\frac{3}{7}$.



(a) Find the K.E. lost in the impact.

(b) Find the magnitude of the impulse exerted by A on B.

Solution:



Motion $\perp$ to AB unchanged

$\Rightarrow w = 5\ ,\quad y = 2\sqrt{3}$

For motion parallel to AB

CLM $\rightarrow +$ $\quad 2m \times 5 - 3m \times 2 = 2mx + 3mz$

$\Rightarrow \quad 2x + 3z = 4 \qquad\qquad \mathbf{I}$

NEL $\quad e = \frac{3}{7} = \frac{z-x}{5+2}$

$\Rightarrow \quad z - x = 3 \qquad\qquad \mathbf{II}$

$\mathbf{I} + 2\mathbf{II} \quad 5z = 10 \quad \Rightarrow \quad z = 2 \Rightarrow x = -1$

(a) $\quad |\underline{\mathbf{v}}_A| = \sqrt{w^2 + x^2} = \sqrt{25+1} = \sqrt{26}$

and $\quad |\underline{\mathbf{v}}_B| = \sqrt{y^2 + z^2} = \sqrt{12+4} = 4$

K.E. lost $= (0{\cdot}5 \times 2m \times 50 + 0{\cdot}5 \times 3m \times 16) - (0{\cdot}5 \times 2m \times 26 + 0{\cdot}5 \times 3m \times 16)$

$= 24m$ J

(b) Impulse of A on B, considering B

$\rightarrow +\quad$ I $= mv - mu = 3m \times 2 - 3m(-2) = 12m$ Ns.

Angle of deflection

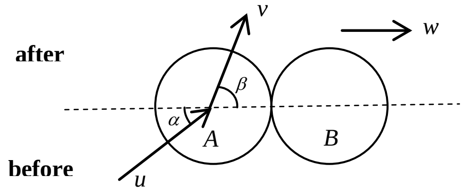
Example: Two identical smooth snooker balls, A and B, are free to move on a horizontal table. A is moving with speed u and collides with B which is stationary. Immediately before the collision A is moving at an angle of α with the line of centres, and immediately after the velocity of A makes an angle β with the line of centres. The coefficient of restitution is e.

(a) Show that $\tan\beta = \frac{2\tan\alpha}{1-e}$.

(b) The collision deflects A through an angle θ. Find $\tan\theta$.

Solution:

(a) Motion $\perp$ to line of centres unchanged



$\Rightarrow$ $v\sin\beta = u\sin\alpha$ **I**

and B moves along line of centres

For motion parallel to line of centres

CLM $mu\cos\alpha = mv\cos\beta + mw$

$\Rightarrow$ $w = u\cos\alpha - v\cos\beta$ **II**

NEL $e = \frac{w - v\cos\beta}{u\cos\alpha} = \frac{u\cos\alpha - 2v\cos\beta}{u\cos\alpha}$ using **II**

$\Rightarrow$ $2v\cos\beta = u\cos\alpha(1-e)$ **III**

I ÷ III $\frac{\sin\beta}{2\cos\beta} = \frac{\sin\alpha}{\cos\alpha(1-e)}$ $\Rightarrow$ $\tan\beta = \frac{2\tan\alpha}{1-e}$ Q.E.D.

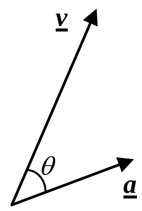
(b) A is deflected through $\theta = \beta - \alpha$

$\Rightarrow$ $\tan\theta = \tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}$

$\Rightarrow$ $\tan\theta = \frac{\frac{2\tan\alpha}{1-e} - \tan\alpha}{1+\frac{2\tan\alpha}{1-e}\tan\alpha} = \frac{(1+e)\tan\alpha}{1-e+2\tan^2\alpha}$

Resolving a velocity in the direction of a given vector

Example: Find the resolved part of $\underline{\boldsymbol{v}} = \begin{pmatrix}2\\5\end{pmatrix}$ m s^{-1} in the direction of the vector $\underline{\boldsymbol{a}} = \begin{pmatrix}3\\1\end{pmatrix}$.



Solution: The resolved part of $\underline{\boldsymbol{v}}$ in the direction of $\underline{\boldsymbol{a}}$ is $v\cos\theta$

But $\underline{\boldsymbol{v}}\cdot\underline{\boldsymbol{a}} = va\cos\theta$ $\Rightarrow$ $v\cos\theta = \frac{\underline{\boldsymbol{v}}.\underline{\boldsymbol{a}}}{a}$

$\underline{\boldsymbol{v}}\cdot\underline{\boldsymbol{a}} = \begin{pmatrix}2\\5\end{pmatrix}\cdot\begin{pmatrix}3\\1\end{pmatrix} = 11$, and $a = \sqrt{3^2+1^2} = \sqrt{10}$

$\Rightarrow$ resolved part of $\underline{\boldsymbol{v}}$ in the direction of $\underline{\boldsymbol{a}}$ is $\frac{\underline{\boldsymbol{v}}.\underline{\boldsymbol{a}}}{a} = \frac{11}{\sqrt{10}} = 3{\cdot}48$ to 3 S.F.

Oblique collisions using vectors

Example: Two identical smooth spheres, A and B both of mass 2 kg, are moving on a horizontal plane, with velocities $\underline{\boldsymbol{u}}_A = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\underline{\boldsymbol{u}}_B = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

They collide and A moves off with velocity $\underline{\boldsymbol{v}}_A = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

(a) Find the impulse acting on A.

(b) Find the velocity of B immediately after impact.

(c) Find the coefficient of restitution.

Solution: (a) $\underline{\boldsymbol{I}}_A = 2\,\underline{\boldsymbol{v}}_A - 2\,\underline{\boldsymbol{u}}_A = 2\begin{pmatrix} 1 \\ -3 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ Ns

(b) $\underline{\boldsymbol{I}}_B = -\,\underline{\boldsymbol{I}}_A = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 2\,\underline{\boldsymbol{v}}_B - 2\,\underline{\boldsymbol{u}}_B = 2\,\underline{\boldsymbol{v}}_B - 2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

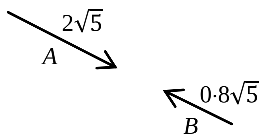
$\Rightarrow$ $\underline{\boldsymbol{v}}_B = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ m s^{-1}.

(c) The impulse must be along the line of centres, so the line of centres is parallel to $\underline{\boldsymbol{I}}_B = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, which has length $2\sqrt{5}$

Components along the line of centres

before A $\begin{pmatrix} 3 \\ -4 \end{pmatrix} . \begin{pmatrix} 4 \\ -2 \end{pmatrix} \times \frac{1}{2\sqrt{5}} = 2\sqrt{5}$

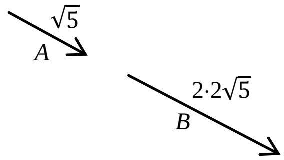
B $\begin{pmatrix} -1 \\ 2 \end{pmatrix} . \begin{pmatrix} 4 \\ -2 \end{pmatrix} \times \frac{1}{2\sqrt{5}} = -0 \cdot 8\sqrt{5}$



$\Rightarrow$ speed of approach $= 2{\cdot}8\sqrt{5}$

after A $\begin{pmatrix} 1 \\ -3 \end{pmatrix} . \begin{pmatrix} 4 \\ -2 \end{pmatrix} \times \frac{1}{2\sqrt{5}} = \sqrt{5}$

B $\begin{pmatrix} 3 \\ -5 \end{pmatrix} . \begin{pmatrix} 4 \\ -2 \end{pmatrix} \times \frac{1}{2\sqrt{5}} = 2 \cdot 2\sqrt{5}$



$\Rightarrow$ speed of separation $= 1{\cdot}2\sqrt{5}$

$\Rightarrow$ $e = \dfrac{\text{speed of separation}}{\text{speed of approach}} = \dfrac{1{\cdot}2\sqrt{5}}{2{\cdot}8\sqrt{5}} = \dfrac{3}{7}$.

3 Resisted motion in a straight line

Acceleration – reminder

Acceleration is $\frac{d^2x}{dt^2} = \ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx}$.

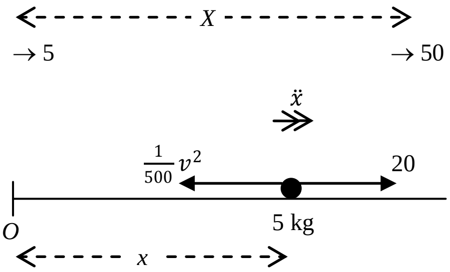
You should be ready to use any one of these – **all** should be measured in the direction of x increasing.

Driving force and air resistance

Example: A particle of mass 5 kg is projected along a smooth horizontal surface with a speed of 5 m s^{-1} in a straight line. There is a constant force of 20 N in the direction of the initial velocity. When the particle is moving at a speed of v m s^{-1}, the air resistance is $\frac{1}{500}v^2$ N.

(*a*) Find how long it takes to acquire a speed of 50 m s^{-1}.

(*b*) Find the distance travelled in gaining a speed of 50 m s^{-1}.



Solution:

(*a*) $R \rightarrow$, N2L $\quad 20 - \frac{1}{500}v^2 = 5\ddot{x}$

In this case we are dealing with speed and time

and so use $20 - \frac{1}{500}v^2 = 5\frac{dv}{dt}$

$$\Rightarrow \int_0^T dt = \int_5^{50} \frac{2500}{10000 - v^2}\,dv = \int_5^{50} \frac{\frac{25}{2}}{100 - v} + \frac{\frac{25}{2}}{100 + v}\,dv$$

$$[\,t\,]_0^T = \left[\frac{25}{2}\ln\left(\frac{100+v}{100-v}\right)\right]_5^{50} \Rightarrow T = \frac{25}{2}\ln\left(\frac{150}{50}\right) = 12\cdot5\ln 3 = 13\cdot7 \text{ seconds}$$

(*b*) In this case we are dealing with speed and distance and use

$R \rightarrow$, N2L $\quad 20 - \frac{1}{500}v^2 = 5v\frac{dv}{dx}$

$$\Rightarrow \int_0^X dx = \int_5^{50} \frac{2500v}{10000 - v^2}\,dv = [\,-1250\ln(10000 - v^2)\,]_5^{50}$$

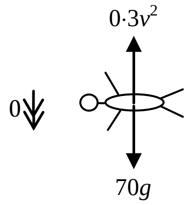
$$\Rightarrow X = -1250\ln\left(\frac{7500}{9975}\right) = 356{\cdot}47\ldots = 356 \text{ m to 3 S.F.}$$

Vertical motion

Terminal speed

Note that terminal speed occurs when the acceleration is zero.

Example: A man of mass 70 kg is falling freely after jumping from an aircraft. The air resistance is $0{\cdot}3v^2$, where v m s^{-1} is his speed. Find his terminal speed.



Solution: For terminal speed, the acceleration is zero,

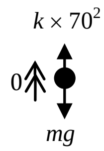
$R\downarrow \quad \Rightarrow \quad 70g - 0{\cdot}3v^2 = 0$

$\Rightarrow \quad v = 47.819\ldots = 48$ m s^{-1} to 2 S.F.

Resistance proportional to square of velocity

Example: A ball is projected vertically upwards with a speed of 49 m s^{-1} from the top of a cliff, which is 180 metres above the sea. As the ball comes down it just misses the cliff. In free fall the ball would have a terminal speed of 70 m s^{-1}, and the air resistance is proportional to the square of the velocity, $R = kv^2$, where k is a positive constant.

Find the greatest height of the ball, and find its speed when it falls into the sea.



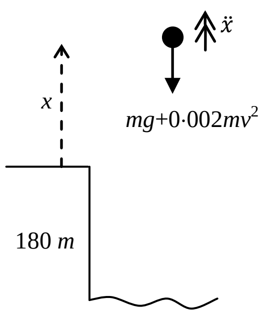
Solution: Measure x from top of cliff

taking $\uparrow$ as the positive direction for x, $\dot{x}$ and $\ddot{x}$

For terminal speed $\quad v = 70, \ \ddot{x} = 0,$

$R\uparrow \quad mg = k \times 70^2 \ \Rightarrow \ k = \frac{9.8}{4900}m = 0{\cdot}002m$

Upwards motion, greatest height



$R\uparrow \quad -mg - 0{\cdot}002mv^2 = m\ddot{x}$

$\Rightarrow \quad -0{\cdot}002(4900 + v^2) = \ddot{x}$

Here we are dealing with speed and distance

use $\quad -0{\cdot}002(4900 + v^2) = v\frac{dv}{dx}$

$\Rightarrow \quad \int_0^X dx = \int_{49}^{0} \frac{-500v}{4900 + v^2}\,dv$

$\Rightarrow \quad X = \left[-250\ln(4900 + v^2)\right]_{49}^{0} = -250\ln\frac{4900}{7301}$

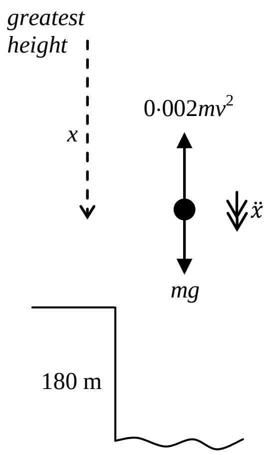
$= 250\ln 1{\cdot}49 = 99{\cdot}6940\ldots$

$\Rightarrow$ greatest height = 100 m to 2 S.F.

Downwards motion, ***new diagram***

Taking ↓ as the positive direction for x, $\dot{x}$ and $\ddot{x}$,

and measuring x from the greatest height



$R\downarrow$ using $\quad mg - 0{\cdot}002mv^2 = mv\,\dfrac{dv}{dx}$

$\Rightarrow \qquad \displaystyle\int dx = \int \frac{500v}{4900 - v^2}\,dv$

$\Rightarrow \qquad x = -250\ln(4900 - v^2) + c$

$\qquad\quad x = 0,\ v = 0 \quad \Rightarrow \quad c = \ln 4900$

$\Rightarrow \qquad x = -250\ln\left(\frac{4900 - v^2}{4900}\right)$

$\Rightarrow \qquad e^{-x}/_{250} = \frac{4900 - v^2}{4900}$

$\Rightarrow \qquad v = 4900\left(1 - e^{-x}/_{250}\right)$

$\Rightarrow \qquad v = 70\sqrt{1 - e^{-0{\cdot}004x}} = 57{\cdot}439\ldots$

when $x = 180 + 250\ln 1{\cdot}49$ from the greatest height

$\Rightarrow$ ball falls into the sea at a speed of 57 m s^{-1} to 2 S.F.

Resistance proportional to velocity

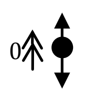
In the case where the resistance force ***R* = *kv*²** **(*k* > 0)**, *R* is always positive, and therefore different diagrams must be drawn for ‘forward’ and ‘backward’ movement, as *R* must always oppose motion.

But when the resistance force ***R* = *kv*** **(*k* > 0)**, *R* is positive when the velocity is positive and negative when the velocity is negative. This allows us to draw one diagram whether the particle is moving ‘forwards’ or ‘backwards’, as in the following example.

Example: A ball is projected vertically upwards with a speed of 49 m s^{-1} from the top of a cliff, which is 180 metres above the sea. As the ball comes down it just misses the cliff. In free fall the ball would have a terminal speed of 70 m s^{-1}, and the air resistance is proportional to the velocity, $R = kv$, where k is a positive constant.

Find the greatest height of the ball, and show that its speed is between 49 m s^{-1} and 50 m s^{-1} when it falls into the sea.

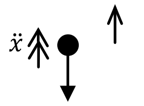
Solution: For terminal speed $v = 70$, $\ddot{x} = 0$,



$R \downarrow \quad mg = k \times 70 \;\Rightarrow\; k = \frac{9.8}{70}m = \frac{7}{50}\,m$

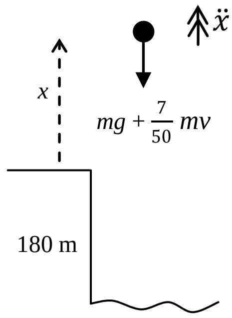
taking $\uparrow$ as the positive direction for x, $\dot{x}$ (= v) and $\ddot{x}$

Note that when the particle is moving upwards, v is positive and so the resistance, $\frac{7}{50}\,mv$, is downwards, *but* when the particle is moving downwards v is negative and so the resistance, $\frac{7}{50}\,mv$, is upwards.



Thus the same diagram can be used for *both* up and down.

Measuring x from the top of the cliff



$R \uparrow \quad -mg - \frac{7}{50}\,mv = m\,\ddot{x}$

$\Rightarrow \quad -\frac{7}{50}\,(70 + v) = v\,\frac{dv}{dx}$

$\Rightarrow \quad \int_0^X dx = \int_{49}^V \frac{-\frac{50}{7}v}{70+v}\,dv = \int_{49}^V \frac{-\frac{50}{7}(v + 70 - 70)}{70+v}\,dv$

$\Rightarrow \quad \int_0^X dx = \frac{50}{7}\int_{49}^V -1 + \frac{70}{70+v}\,dv$

$\Rightarrow \quad \left[x\ \right]_0^X = \left[\frac{50}{7}(-v + 70\ln(70+v))\ \right]_{49}^V$

$\Rightarrow \quad X = \frac{50}{7}\left(-V + 49 + 70\ln\left(\frac{70+V}{119}\right)\right)$

At a speed of 49 m s^{-1} downwards $V = -49$, $\Rightarrow X = \frac{50}{7}\left(49 + 49 + 70\ln\left(\frac{70-49}{119}\right)\right) = -167{\cdot}3\ldots > -180$,

and when $V = -50$, $X = \frac{50}{7}\left(49 + 50 + 70\ln\left(\frac{70-50}{119}\right)\right) = -184{\cdot}55\ldots < -180$

$\Rightarrow$ when the ball falls into the sea its speed is between 49 m s^{-1} and 50 m s^{-1}.

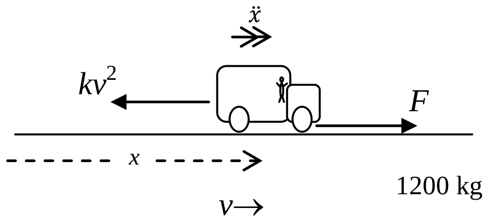
Power and resisting force

If an engine is working at a constant rate, P, then the power, $P = Fv$. As v varies so will F.

Example: A car of mass 1200 kg is travelling on a straight horizontal road, with its engine working at a constant rate of 25 kW. The resistance to motion is proportional to the square of the velocity, and the greatest speed the car can maintain is 50 m s^{-1}.

Show that $125000 - v^3 = 6000\, v^2 \frac{dv}{dx}$, where v m s^{-1} is the velocity of the car when x metres is its displacement from a fixed point on the road. Hence find the distance travelled by the car in increasing its speed from 30 m s^{-1} to 45 m s^{-1}.

Solution: Power $P = 25000$



$\Rightarrow \quad F = \frac{P}{v} = \frac{25000}{v}$

If the greatest possible speed is 50 m s^{-1}

$\frac{25000}{50} = k \times 50^2 \Rightarrow k = 0{\cdot}2$

$R \rightarrow$, N2L $\quad \frac{25000}{v} - 0{\cdot}2v^2 = 1200\, \ddot{x} = 1200\, v\frac{dv}{dx}$

$\Rightarrow \quad 125000 - v^3 = 6000\, v^2 \frac{dv}{dx}$ Q.E.D.

$$\Rightarrow \quad \int_0^X dx = \int_{30}^{45} \frac{6000v^2}{125000 - v^3}\, dv$$

$$\Rightarrow \quad \left[\, x \,\right]_0^X = \left[-2000 \ln(125000 - v^3)\right]_{30}^{45}$$

$\Rightarrow \quad X = 2000 \ln\left(\frac{98000}{33875}\right) = 2124{\cdot}58\ldots = 2125$ m to the nearest metre.

4 Damped and forced harmonic motion

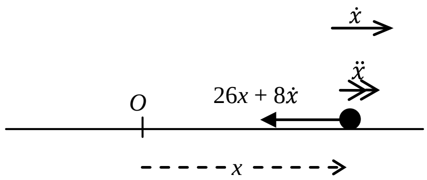
Damped harmonic motion

A particle executing S.H.M. and also subject to a resistance force which is proportional to the speed. Other types of resistance forces are beyond the scope of this course.

Example 1*:* A particle, P, of mass 2 kg is moving in a straight line. Its distance from a fixed point, O, on the line after t seconds is x metres. A force of 26 N acts on the particle towards O, and a resistance force of magnitude $8v$ N also acts on the particle.

When $t = 0$, $x = 1{\cdot}5$ m and $\dot{x} = 9$ m s^{-1}.

Solution: take $\rightarrow$ as the positive direction for x, $\dot{x}$ and $\ddot{x}$.



Note that when the particle is moving right, $\dot{x}$ is positive and so the resistance is to the left, *but* when the particle is moving left $\dot{x}$ is negative and so the resistance is to the right.

Thus the same diagram can be used when the particle is moving *both* left and right.

$R \rightarrow$, N2L $\quad -26x - 8\dot{x} = 2\ddot{x}$

$\Rightarrow \quad \ddot{x} + 4\dot{x} + 13x = 0$

A.E. is $m^2 + 4m + 13 = 0$

$\Rightarrow \quad (m + 2)^2 + 9 = 0 \quad \Rightarrow \quad m = -2 \pm 3i$

$\Rightarrow \quad$ G.S. is $x = e^{-2t}(A\cos 3t + B\sin 3t)$

$x = 1{\cdot}5$ when $t = 0$, $\quad \Rightarrow \quad A = 1{\cdot}5$

$\dot{x} = -2e^{-2t}(A\cos 3t + B\sin 3t) + e^{-2t}(-3A\sin 3t + 3B\cos 3t)$

and $\quad \dot{x} = 9$ when $t = 0 \quad \Rightarrow \quad 9 = -2A + 3B \quad \Rightarrow \quad B = 4$

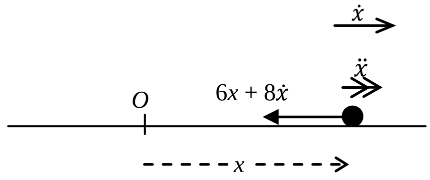
$\Rightarrow \quad x = e^{-2t}(1{\cdot}5\cos 3t + 4\sin 3t)$

In this case the particle continues to oscillate with ever decreasing amplitude. This is called *light damping*.

Example 2*:* A particle, *P*, of mass 2 kg is moving in a straight line. Its distance from a fixed point, *O*, on the line after *t* seconds is *x* metres. A force of 6 N acts on the particle towards *O*, and a resistance force of magnitude 8*v* N also acts on the particle.

When $t = 0$, $x = 3$ m and $\dot{x} = 9$ m s^{-1}.

Solution: take $\rightarrow$ as the positive direction for x, $\dot{x}$ and $\ddot{x}$.



$R \rightarrow$, N2L $\quad -6x - 8\dot{x} = 2\ddot{x}$

$\Rightarrow \quad \ddot{x} + 4\dot{x} + 3x = 0$

A.E. is $m^2 + 4m + 3 = 0$

$\Rightarrow \quad (m + 1)(m + 3) = 0 \quad \Rightarrow \quad m = -1$ or -3

$\Rightarrow \quad$ G.S. is $x = Ae^{-t} + Be^{-3t}$

$x = 3$ when $t = 0$, $\Rightarrow A + B = 3$ **I**

$\dot{x} = -Ae^{-t} - 3Be^{-3t}$

and $\dot{x} = 9$ when $t = 0 \quad \Rightarrow \quad 9 = -A - 3B$ **II**

I + II $\Rightarrow B = -6$, $\Rightarrow A = 9$

$\Rightarrow \quad x = 9e^{-t} - 3e^{-3t}$

In this case the particle does **not** oscillate but just goes ‘gludge’.
This is called *heavy damping*.

Light, critical and heavy damping

The differential equation for damped S.H.M. will always be of the form

$$\ddot{x} + k\dot{x} + lx = 0, \qquad \text{where } k > 0 \text{ and } l > 0$$

think about it!

If $k^2 - 4l < 0$, we have complex roots, like example 1, and we have *light damping*.

If $k^2 - 4l = 0$, we have equal real roots and we have *critical damping*.

If $k^2 - 4l > 0$, we have distinct real roots, like example 2, and we have *heavy damping*.

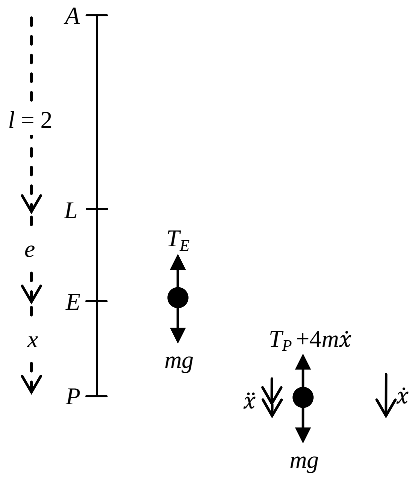
Damped harmonic motion on a vertical elastic string

As always, you must measure *x* from the equilibrium position.

Example: A particle of mass m is suspended at the lower end of a vertical elastic string, of which the upper end is fixed at A. The modulus of elasticity is $40m$ N and the natural length of the string is 2 metres. A resistance force of magnitude $4mv$ acts on the particle, where v is the speed at time t.

The particle is pulled down a distance of 0·5 metres below the equilibrium position and released. Find an expression for x, the distance below the equilibrium position, in terms of t, and describe the damping.

Solution: As previously, if we mark the resistance force as $4m\dot{x}$ (as shown) it will always be in the direction opposite to motion.



At the equilibrium position, E

$R\downarrow \quad mg = T_E = \frac{\lambda e}{l}$

and at P, x below the equilibrium position,

$R\downarrow \quad mg - T_P - 4m\dot{x} = m\ddot{x}$

$\Rightarrow \quad mg - \frac{\lambda(e+x)}{l} - 4m\dot{x} = m\ddot{x}$

$\Rightarrow \quad \ddot{x} + 4\dot{x} + \frac{40m}{2m}x = 0$ since $mg = \frac{\lambda e}{l}$ and $\lambda = 40$ m

$\Rightarrow \quad \ddot{x} + 4\dot{x} + 20x = 0$

A.E. $\quad m^2 + 4m + 20 = 0 \quad \Rightarrow \quad m = -2 \pm 4i$

G.S. $\quad x = e^{-2t}(A\cos 4t + B\sin 4t)$

When $t = 0$, $x = 0{\cdot}5$ and $\dot{x} = 0$,

$\Rightarrow \quad 0{\cdot}5 = A$

$\dot{x} = -2e^{-2t}(A\cos 4t + B\sin 4t) + e^{-2t}(-4A\sin 4t + 4B\cos 4t)$

$\Rightarrow \quad 0 = -2A + 4B \quad \Rightarrow \quad B = 0{\cdot}25$

$\Rightarrow \quad x = e^{-2t}(0{\cdot}5\cos 4t + 0{\cdot}25\sin 4t)$

which is light damping.

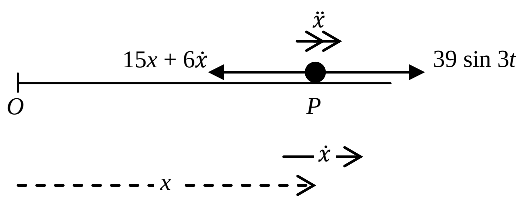
Forced harmonic motion

This is similar to the previous section, with an additional periodic force.

Example: A particle P of mass 3 kg is moving on the x-axis. At time t the displacement of P from the origin O is x metres and the speed of P is v m s^{-1}. Three forces act on P – a force of $15x$ N towards O, a resistance to the motion of magnitude $6v$ N and a force $39 \sin 3t$ N in the direction OP.

Find x as a function of t, and describe the motion when t becomes large.

Solution: $R \rightarrow$, N2L



$\Rightarrow \quad 39 \sin 3t - 15x - 6\dot{x} = 3\ddot{x}$

$\Rightarrow \quad \ddot{x} + 2\dot{x} + 5x = 13 \sin 3t$

A.E. $\quad m^2 + 2m + 5 = 0$

$\Rightarrow \quad (m + 1)^2 = -4 \quad \Rightarrow \quad m = -1 \pm 2i$

C.F. $\quad x = e^{-t}(A \cos 2t + B \sin 2t)$

For P.I. try $\quad x = C \sin 3t + D \cos 3t$

$\Rightarrow \quad \dot{x} = 3C \cos 3t - 3D \sin 3t$

$\Rightarrow \quad \ddot{x} = -9C \sin 3t - 9D \cos 3t$

$\Rightarrow \quad -9C \sin 3t - 9D \cos 3t + 6C \cos 3t - 6D \sin 3t + 5C \sin 3t + 5D \cos 3t = 13 \sin 3t$

$\Rightarrow \quad -9C - 6D + 5C = 13 \quad \Rightarrow \quad -4C - 6D = 13$

and $\quad -9D + 6C + 5D = 0 \quad \Rightarrow \quad 3C - 2D = 0$

$\Rightarrow \quad C = -1$ and $D = -1{\cdot}5$

G.S. $\quad x = e^{-t}(A \cos 2t + B \sin 2t) - \sin 3t - 1{\cdot}5 \cos 3t$

as t becomes large, $e^{-t} \rightarrow 0$,

$\Rightarrow \quad x \approx -\sin 3t - 1{\cdot}5 \cos 3t$,

which is S.H.M with period $\frac{2\pi}{3}$ and amplitude $\sqrt{1 + 1 \cdot 5^2} = \sqrt{3 \cdot 25}$.

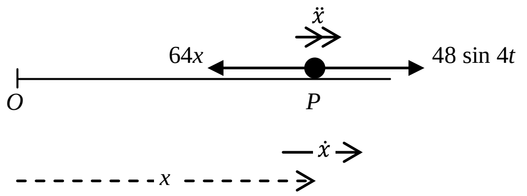
Resonance

When a particle is undergoing S.H.M. with period $\frac{2\pi}{\omega}$, and a force with the same period $\frac{2\pi}{\omega}$ is applied, resonance occurs.

Example: A particle P of mass 3 kg is moving on the x-axis. At time t the displacement of P from the origin O is x metres. Two forces act on P – a force of $64x$ N towards O (the force needed for S.H.M) and a force $48 \sin 4t$ N in the direction OP.

Find x as a function of t, and describe the motion when t becomes large.

Solution: $R \rightarrow$, N2L $\quad 48 \sin 4t - 64x = 4\ddot{x}$



$\Rightarrow \quad \ddot{x} + 16x = 12 \sin 4t$

A.E. $\quad m^2 + 16m = 0$

$\Rightarrow \quad m = \pm 4i$

C.F. $\quad x = A \sin 4t + B \cos 4t$

since $\sin 4t$ occurs in the C.F., for the P.I. we try

$x = Ct \cos 4t$ (generally we should try ($Ct \cos 4t + Dt \sin 4t$), but $Ct \cos 4t$ works here)

$\Rightarrow \quad \dot{x} = C \cos 4t - 4Ct \sin 4t$

and $\quad \ddot{x} = -8C \sin 4t - 16Ct \cos 4t$

$\Rightarrow \quad -8C \sin 4t - 16Ct \cos 4t + 16Ct \cos 4t = 12 \sin 4t$

$\Rightarrow \quad C = -1{\cdot}5$

G.S. $\quad x = A \sin 4t + B \cos 4t - 1{\cdot}5t \sin 4t.$

As t increases, the term $-1{\cdot}5t \sin 4t$ dominates whatever the values of A and B, and we have oscillations with ever increasing amplitude – this is resonance.

5 Stable and unstable equilibrium

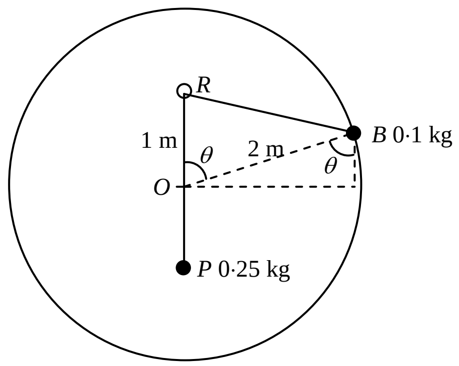
Potential energy of a system

To investigate the stability of a system:

1. Choose a **fixed** point or **fixed** level from which to measure the potential energy¸ V.
2. Calculate V relative to the **fixed** point or **fixed** level. P.E. is positive for masses **above** the fixed point, and negative for masses **below** the fixed point.
3. Writing 'V + constant' gives the potential energy of the system relative to any **fixed** point or **fixed** level. This is similar to writing '+ c' when integrating.
4. Differentiate V (probably) to find maxima and minima.
5. Stable positions of equilibrium occur at minima of V, and unstable positions of equilibrium occur at maxima of V.

Example: A smooth circular wire of radius 2 metres is fixed in a vertical plane. A bead, B, of mass 0·1 kg is threaded onto the wire. A small smooth ring, R, is fixed 1 metre above the centre of the wire. An inextensible string of length 4 metres, with one end attached to the bead, passes through the ring; a particle, P, of mass 0·25 kg, attached to the other end of the string, and hangs vertically below the ring. Find the positions of equilibrium and investigate their stability.

Solution: Calculate the potential energy of the system relative to the centre of the circle, O.



The bead, B, is $2\cos\theta$ above O,

$\Rightarrow$ P.E. of bead is $0{\cdot}1\,g \times 2\cos\theta = 0{\cdot}2\,g\cos\theta$

Note that if $\theta > 90^o$, B is below O and cos θ is negative, so P.E. is negative as it should be.

For the P.E. of the particle, P, we need the length RB.

Cosine rule $\Rightarrow RB^2 = 1^2 + 2^2 - 2 \times 1 \times 2 \times \cos\theta$

$\Rightarrow \qquad RB = \sqrt{5 - 4\cos\theta}$

Length of string is 4 m $\Rightarrow \quad OP = 4 - 1 - \sqrt{5 - 4\cos\theta}$

$\Rightarrow \qquad$ P.E. of particle is $-0{\cdot}25 \times g \times (3 - \sqrt{5 - 4\cos\theta})$ negative as P is below O

Note that if P is above O then OP is negative and the P.E. is positive, as it should be.

Thus the P.E. of the system is

$$V = 0{\cdot}2\,g\cos\theta - 0{\cdot}25 \times g \times (3 - \sqrt{5 - 4\cos\theta})$$

$$\Rightarrow \qquad V = 0{\cdot}2\,g\cos\theta + 0{\cdot}25 \times g \times \sqrt{5 - 4\cos\theta} + \text{constant}$$

*Note that the 'constant' allows the P.E. to be measured from **any** fixed level.*

To find the positions of equilibrium

$$\frac{dV}{d\theta} = -0{\cdot}2\, g \sin\theta \;+\; 0{\cdot}25 \times g \times \frac{1}{2}(5-4\cos\theta)^{\frac{-1}{2}} \times 4\sin\theta$$

$$\frac{dV}{d\theta} = 0 \;\Rightarrow\; g\sin\theta\left(-0{\cdot}2 + \frac{1}{2}(5-4\cos\theta)^{\frac{-1}{2}}\right)$$

$\Rightarrow$ $\sin\theta = 0$ or $5 - 4\cos\theta = \frac{25}{4} \Rightarrow \cos\theta = \frac{-5}{16}$

$\Rightarrow$ $\theta = 0^{\circ}, 180^{\circ}, 108{\cdot}2^{\circ}$ or $251{\cdot}8^{\circ}$

To investigate the stability

$$\frac{d^2V}{d\theta^2} = g\cos\theta\left(-0{\cdot}2 + \frac{1}{2}(5-4\cos\theta)^{\frac{-1}{2}}\right) + g\sin\theta \times \frac{-1}{4}(5-4\cos\theta)^{\frac{-3}{2}} \times 4\sin\theta$$

$\theta = 0 \Rightarrow \frac{d^2V}{d\theta^2} = g \times (-0{\cdot}2 + 0{\cdot}5) > 0 \Rightarrow$ min V $\Rightarrow$ STABLE

$\theta = 180 \Rightarrow \frac{d^2V}{d\theta^2} = -g \times \left(-0{\cdot}2 + \frac{1}{2} \times \frac{1}{3}\right) > 0 \Rightarrow$ min V $\Rightarrow$ STABLE

$\theta = 108.2 \Rightarrow \frac{d^2V}{d\theta^2} = g \times (0 - 0{\cdot}0577\ldots) < 0 \Rightarrow$ max V $\Rightarrow$ UNSTABLE

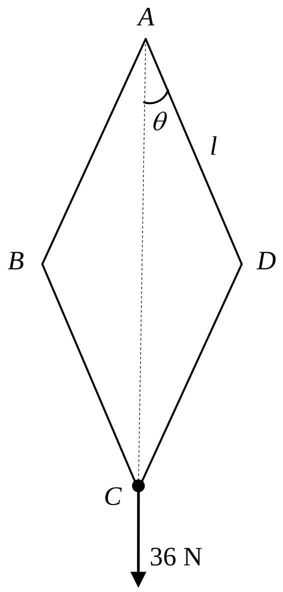
$\theta = 251.8 \Rightarrow \frac{d^2V}{d\theta^2} = g \times (0 - 0{\cdot}0577\ldots) < 0 \Rightarrow$ max V $\Rightarrow$ UNSTABLE

Elastic strings or springs and P.E.

When an elastic string is stretched, or an elastic spring is stretched or compressed, the elastic potential energy, E.P.E., is always taken as positive. *If the stretch (or compression) is decreased then energy is released*.

Example: A framework consists of 4 identical light rods of length l m. The rods are smoothly joined at their ends. The framework is suspended from a fixed point, A, and a weight of 36 N is attached at C. B and D are connected by a light elastic spring of natural length l m, and modulus of elasticity 65 N.

Show that the framework can rest in equilibrium when each rod makes and angle of $\sin^{-1}\left(\frac{5}{13}\right)$ and investigate the stability.



Solution: Take A as the fixed level from which P.E. is measured. Let $\angle CAD = \theta$.

P.E. of the weight is $-2l\cos\theta \times 36$

$BD = 2\times l\sin\theta$

$\Rightarrow$ E.P.E. $= \frac{1}{2}\times 65\times\frac{(l-2l\sin\theta)^2}{l}$

$\Rightarrow$ $V = \frac{1}{2}\times 65\times\frac{(l-2l\sin\theta)^2}{l} - 72l\cos\theta$ (+ const)

$\Rightarrow$ $\frac{dV}{d\theta} = \frac{1}{2}\times 65\times\frac{2\times(l-2l\sin\theta)\times(-2l\cos\theta)}{l} + 72l\sin\theta$

$= -130l(1-2\sin\theta)\cos\theta + 72l\sin\theta$

When $\theta = \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{12}{13}\right)$ 5, 12, 13 triangle

$\frac{dV}{d\theta} = -130l\times\frac{3}{13}\times\frac{12}{13} + 72l\times\frac{5}{13} = 0$

$\Rightarrow$ Equilibrium when $\theta = \sin^{-1}\left(\frac{5}{13}\right)$

$\frac{dV}{d\theta} = -130l\cos\theta + 130l\sin 2\theta + 72l\sin\theta$

$\frac{d^2V}{d\theta^2} = 130l\sin\theta + 260l\cos 2\theta + 72l\cos\theta$

$= 130l\sin\theta + 260l(1-2\sin^2\theta) + 72l\cos\theta$

$= 130l\times\frac{5}{13} + 260l\times\left(1-2\times\left(\frac{5}{13}\right)^2\right) + 72l\times\frac{12}{13}$ = a positive number

$\Rightarrow$ minimum of V, $\Rightarrow$ equilibrium is stable.

I did try putting AC = 2x, instead of using θ – not a good idea!!

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