Pure
Further Mathematics 2

Revision Notes

November 2014

Further Pure 2

1 Inequalities

Algebraic solutions

Remember that if you multiply both sides of an inequality by a negative number, you must turn the inequality sign round: $2x > 3 \Rightarrow -2x < -3$.

A difficulty occurs when multiplying both sides by, for example, $(x - 2)$; this expression is sometimes positive ($x > 2$), sometimes negative ($x < 2$) and sometimes zero ($x = 2$). In this case we multiply both sides by $(x - 2)^2$, which is always positive (provided that $x \neq 2$).

Example 1: Solve the inequality $2x + 3 < \frac{x^2}{x-2}$, $\quad x \neq 2$

Solution: Multiply both sides by $(x - 2)^2$ — we can do this since $(x - 2) \neq 0$

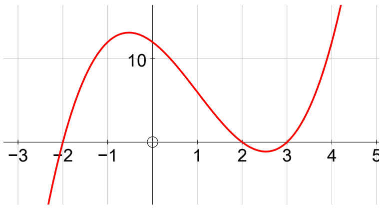
$\Rightarrow \quad (2x + 3)(x - 2)^2 < x^2(x - 2)$ — DO **NOT** MULTIPLY OUT

$\Rightarrow \quad (2x + 3)(x - 2)^2 - x^2(x - 2) < 0$

$\Rightarrow \quad (x - 2)(2x^2 - x - 6 - x^2) < 0$

$\Rightarrow \quad (x - 2)(x - 3)(x + 2) < 0$

$\Rightarrow \quad x < -2, \text{ or } 2 < x < 3$



Note – care is needed when the inequality is $\leq$ or $\geq$.

Example 2: Solve the inequality $\frac{x}{x+1} \geq \frac{2}{x+3}$, $\quad x \neq -1, \; x \neq -3$

Solution: Multiply both sides by $(x + 1)^2(x + 3)^2$ — which cannot be zero

$\Rightarrow \quad x(x + 1)(x + 3)^2 \geq 2(x + 3)(x + 1)^2$ — DO **NOT** MULTIPLY OUT

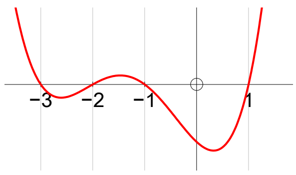
$\Rightarrow \quad x(x + 1)(x + 3)^2 - 2(x + 3)(x + 1)^2 \geq 0$

$\Rightarrow \quad (x + 1)(x + 3)(x^2 + 3x - 2x - 2) \geq 0$

$\Rightarrow \quad (x + 1)(x + 3)(x + 2)(x - 1) \geq 0$

from sketch it looks as though the solution is

$$x \leq -3 \quad \text{or} \quad -2 \leq x \leq -1 \quad \text{or} \quad x \geq 1$$



BUT since $x \neq -1, \; x \neq -3$,

the solution is $\quad x < -3 \quad \text{or} \quad -2 \leq x < -1 \quad \text{or} \quad x \geq 1$

Graphical solutions

Example 1: On the same diagram sketch the graphs of $y = \frac{2x}{x+3}$ and $y = x - 2$.

Use your sketch to solve the inequality $\frac{2x}{x+3} \geq x - 2$

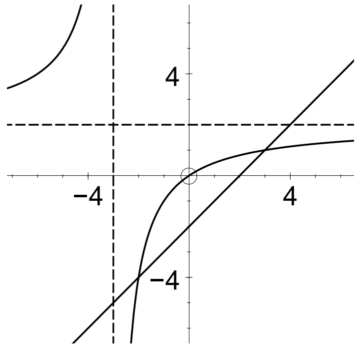
Solution: First find the points of intersection of the two graphs

$\Rightarrow \quad \frac{2x}{x+3} = x - 2$

$\Rightarrow \quad 2x = x^2 + x - 6$

$\Rightarrow \quad 0 = (x-3)(x+2)$

$\Rightarrow \quad x = -2 \text{ or } 3$



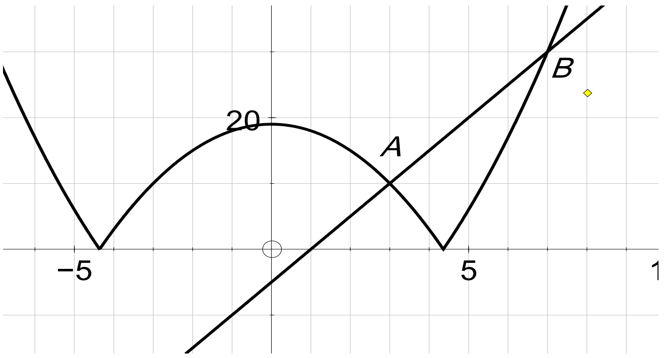
From the sketch we see that

$x < -3$ or $-2 \leq x \leq 3$. Note that $x \neq -3$

For inequalities involving |2*x* – 5| etc., it is often essential to sketch the graphs first.

Example 2: Solve the inequality $|x^2 - 19| < 5(x - 1)$.

Solution: It is essential to sketch the curves first in order to see which solutions are needed.



To find the point A, we need to solve

$-(x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 + 5x - 24 = 0$

$\Rightarrow \quad (x+8)(x-3) = 0 \quad \Rightarrow \quad x = -8 \text{ or } 3$

From the sketch $x \neq -8 \quad \Rightarrow \quad x = 3$

To find the point B, we need to solve

$$+(x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 - 5x - 14 = 0$$

$$\Rightarrow \quad (x-7)(x+2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } 7$$

From the sketch $x \neq -2 \quad \Rightarrow \quad x = 7$

and the solution of $|x^2 - 19| < 5(x-1)$ is $3 < x < 7$

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2 Series – Method of Differences

The trick here is to write each line out in full and see what cancels when you add.

Do not be tempted to work each term out – you will lose the pattern which lets you cancel when adding.

Example 1*:* Write $\frac{1}{r(r+1)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{\mathrm{r(r+1)}} = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{n(n+1)}$.

Solution: $$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

put $r = 1 \Rightarrow \quad \frac{1}{1\times2} = \frac{1}{1} - \frac{1}{2}$

put $r = 2 \Rightarrow \quad \frac{1}{2\times3} = \frac{1}{2} - \frac{1}{3}$

put $r = 3 \Rightarrow \quad \frac{1}{3\times4} = \frac{1}{3} - \frac{1}{4}$

etc.

put $r = n \Rightarrow \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

adding $\Rightarrow \quad \sum_1^n \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Example 2: Write $\frac{2}{r(r+1)(r+2)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{\mathrm{r(r+1)(r+2)}} = \frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \cdots + \frac{1}{n(n+1)(n+2)}$.

Solution: $\quad \frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$

put $r = 1 \quad \Rightarrow \quad \frac{2}{1\times2\times3} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$

put $r = 2 \quad \Rightarrow \quad \frac{2}{2\times3\times4} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

put $r = 3 \quad \Rightarrow \quad \frac{2}{3\times4\times5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

put $r = 4 \quad \Rightarrow \quad \frac{2}{4\times5\times6} = \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$

$\vdots$

etc.

$\vdots$

put $r = n-1 \quad \Rightarrow \quad \frac{2}{(n-1)n(n+1)} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$

put $r = n \quad \Rightarrow \quad \frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

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adding $\Rightarrow \quad \sum_1^n \frac{2}{r(r+1)(r+2)} = \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

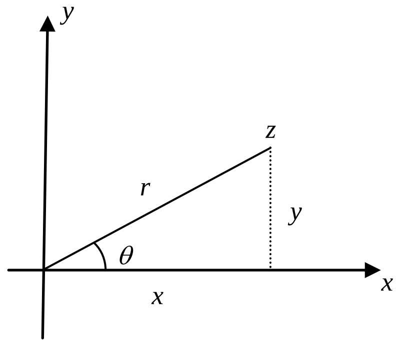
$$= \frac{n^2+3n+2-2n-4+2n+2}{2(n+1)(n+2)}$$

$\Rightarrow \quad \sum_1^n \frac{2}{r(r+1)(r+2)} = \frac{n^2+3n}{2(n+1)(n+2)}$

$\Rightarrow \quad \sum_1^n \frac{1}{r(r+1)(r+2)} = \frac{n^2+3n}{4(n+1)(n+2)}$

3 Complex Numbers

Modulus and Argument



The modulus of $z = x + iy$ is the length of z

$\Rightarrow \quad r = |z| = \sqrt{x^2 + y^2}$

and the argument of z is the angle made by z with the positive x-axis, between $-\pi$ and π.

N.B. $\arg z$ is **not always** equal to $\tan^{-1}\left(\frac{y}{x}\right)$

Properties

$$z = r\cos\theta + i\,r\sin\theta$$

$$|zw| = |z|\,|w|, \quad \text{and} \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$\arg(zw) = \arg z + \arg w, \quad \text{and} \quad \arg\left(\frac{z}{w}\right) = \arg z - \arg w$$

Euler's Relation $e^{i\theta}$

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$$

Example: Express $5e^{\left(\frac{i3\pi}{4}\right)}$ in the form $x + iy$.

Solution: $5e^{\left(\frac{i3\pi}{4}\right)} = 5\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$$= \frac{-5\sqrt{2}}{2} + i\,\frac{5\sqrt{2}}{2}$$

Multiplying and dividing in mod-arg form

$$re^{i\theta} \times se^{i\phi} = rs\,e^{i(\theta+\phi)}$$

$$\equiv \quad (r\cos\theta + i\,r\sin\theta) \times (s\cos\phi + i\,s\sin\phi) = rs\cos(\theta+\phi) + i\,rs\sin(\theta+\phi)$$

and

$$re^{i\theta} \div se^{i\phi} = \frac{r}{s}\,e^{i(\theta-\phi)}$$

$$\equiv (r\cos\theta + i\,r\sin\theta) \div (s\cos\phi + i\,s\sin\phi) = \frac{r}{s}\cos(\theta-\phi) + i\,\frac{r}{s}\sin(\theta-\phi)$$

De Moivre's Theorem

$$\left(re^{i\theta}\right)^n \;=\; r^n e^{in\theta} \quad \equiv \quad (r\cos\theta + i\, r\sin\theta)^n = (r^n\cos n\theta + i\, r^n \sin n\theta)$$

Applications of De Moivre's Theorem

Example: Express $\sin 5\theta$ in terms of $\sin\theta$ only.

Solution: From De Moivre's Theorem we know that

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\,\sin\theta + 10i^2\cos^3\theta\,\sin^2\theta + 10i^3\cos^2\theta\,\sin^3\theta + 5i^4\cos\theta\,\sin^4\theta + i^5\sin^5\theta$$

Equating complex parts

$$\Rightarrow \quad \sin 5\theta = 5\cos^4\theta\,\sin\theta - 10\cos^2\theta\,\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$\boldsymbol{z^n + \frac{1}{z^n} = 2\cos n\theta}$ **and** $\boldsymbol{z^n - \frac{1}{z^n} = 2\, i\sin n\theta}$

$$z = \cos\theta + i\sin\theta$$

$$\Rightarrow \quad z^n = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\ \sin n\theta)$$

and $\quad \frac{1}{z^n} = (\cos\theta - i\sin\theta)^n = (\cos n\theta - i\ \sin n\theta)$

from which we can show that

$$\left(z + \frac{1}{z}\right) = 2\cos\theta \quad \text{and} \quad \left(z - \frac{1}{z}\right) = 2i\sin\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Example: Express $\sin^5\theta$ in terms of $\sin 5\theta$, $\sin 3\theta$ and $\sin\theta$.

Solution: Here we are dealing with $\sin\theta$, so we use

$$(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

$$\Rightarrow \quad 32i\sin^5\theta = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \left(\frac{1}{z^5}\right)$$

$$\Rightarrow \quad 32i\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \quad 32i\sin^5\theta = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$$

$$\Rightarrow \quad \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

n^{th} roots of a complex number

The technique is the same for finding n^{th} roots of any complex number.

Example: Find the 4^{th} roots of $4 + 4i$, and show the roots on an Argand Diagram.

Solution: We need to solve the equation $z^4 = 4 + 4i$

1. Let $z = r\cos\theta + i\,r\sin\theta$

   $\Rightarrow \quad z^4 = r^4(\cos 4\theta + i\sin 4\theta)$

2. $|4 + 4i| = \sqrt{4^2 + 4^2} = \sqrt{32}$ and $\arg(4 + 4i) = \frac{\pi}{4}$

   $\Rightarrow \quad 4 + 4i = \sqrt{32}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3. Then $z^4 = 4 + 4i$

   becomes $r^4(\cos 4\theta + i\sin 4\theta) = \sqrt{32}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

   $= \sqrt{32}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$ adding 2π

   $= \sqrt{32}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$ adding 2π

   $= \sqrt{32}\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right)$ adding 2π

4. $\Rightarrow \quad r^4 = \sqrt{32}$

   and $\quad 4\theta = \frac{\pi}{4}, \ \frac{9\pi}{4}, \ \frac{17\pi}{4}, \ \frac{25\pi}{4}$

   $\Rightarrow \quad r = \sqrt[8]{32} = 1.5422$

   and $\quad \theta = \frac{\pi}{16}, \ \frac{9\pi}{16}, \ \frac{17\pi}{16}, \ \frac{25\pi}{16}$

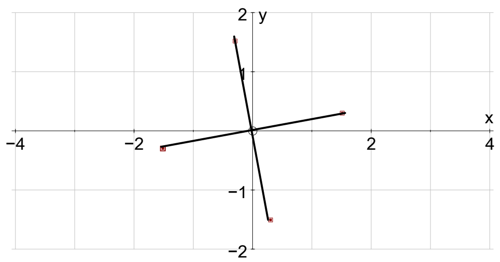
5. $\Rightarrow$ roots are

   $\sqrt[8]{32}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right) = 1.513 + 0.301\,i$

   $\sqrt[8]{32}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right) = -0.301 + 1.513\,i$

   $\sqrt[8]{32}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right) = -1.513 - 0.301\,i$

   $\sqrt[8]{32}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right) = 0.301 - 1.513\,i$



Notice that the roots are symmetrically placed around the origin, and the angle between roots is $\frac{2\pi}{4} = \frac{\pi}{2}$ The angle between the n^{th} roots will always be $\frac{2\pi}{n}$.

For sixth roots the angle between roots will be $\frac{2\pi}{6} = \frac{\pi}{3}$, and so on.

Roots of polynomial equations with real coefficients

1. **Any** polynomial equation with real coefficients,
   $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots a_2 x^2 + a_1 x + a_0 = 0$, …….. (I)
   where all a_i are real, has a complex solution
2. $\Rightarrow$ **any** complex n^{th} degree polynomial can be factorised into n linear factors over the complex numbers
3. If $z = a + ib$ is a root of (I), then its conjugate, $a - ib$ is also a root.
4. By pairing factors with conjugate pairs we can say that any polynomial with real coefficients can be factorised into a combination of linear and quadratic factors over the real numbers.

Example: Given that $3 - 2i$ is a root of $z^3 - 5z^2 + 7z + 13 = 0$

(a) Factorise over the real numbers

(b) Find all three real roots

Solution:

(a) $3 - 2i$ is a root $\Rightarrow$ $3 + 2i$ is also a root

$\Rightarrow$ $(z - (3 - 2i))(z - (3 + 2i)) = (z^2 - 6z + 13)$ is a factor

$\Rightarrow$ $z^3 - 5z^2 + 7z + 13 = (z^2 - 6z + 13)(z + 1)$ by inspection

(b) $\Rightarrow$ roots are $z = 3 - 2i,\ 3 + 2i$ and -1

Loci on an Argand Diagram

Two basic ideas

1. $|z - w|$ is the distance from w to z .
2. $\arg(z - (1 + i))$ is the angle made by the line joining $(1+i)$ to z, with the x-axis.

Example 1:

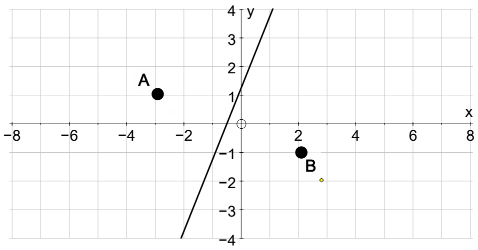
$|z - 2 - i| = 3$ is a circle with centre $(2 + i)$ and radius 3

Example 2:

$|z + 3 - i| = |z - 2 + i|$

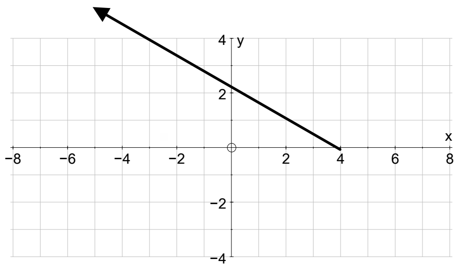
$\Leftrightarrow$ $|z - (-3 + i)| = |z - (2 - i)|$

is the locus of all points which are equidistant from the points
$A\,(-3, 1)$ and $B\,(2, -1)$, and so is the perpendicular bisector of AB.



Example 3:

arg $(z - 4) = \frac{5\pi}{6}$ is a half line, from (4, 0), making an angle of $\frac{5\pi}{6}$ with the x-axis.



Example 4:

$|z - 3| = 2|z + 2i|$ is a circle (Apollonius's circle).

To find its equation, put $z = x + iy$

$\Rightarrow \quad |(x-3) + iy| = 2|x + i(y+2)|$ square both sides

$\Rightarrow \quad (x-3)^2 + y^2 = 4\left(x^2 + (y+2)^2\right)$ leading to

$\Rightarrow \quad 3x^2 + 6x + 3y^2 + 16y + 7 = 0$

$\Rightarrow \quad (x+1)^2 + \left(y + \frac{8}{3}\right)^2 = \frac{52}{9}$

which is a circle with centre $\left(-1, \frac{-8}{3}\right)$, and radius $\frac{2\sqrt{13}}{3}$.

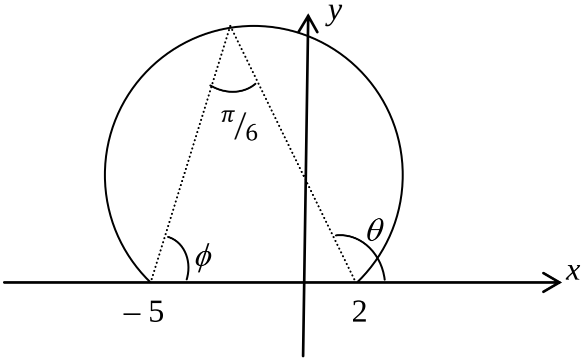
Example 5:

$\arg\left(\frac{z-2}{z+5}\right) = \frac{\pi}{6}$

$\Rightarrow \quad \arg(z-2) - \arg(z+5) = \frac{\pi}{6}$

$\Rightarrow \quad \theta - \phi = \frac{\pi}{6}$

which gives the arc of the circle as shown.

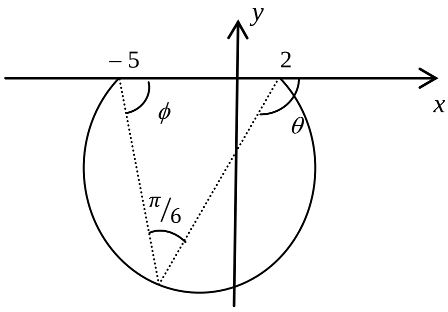


N.B.

The corresponding arc below the x-axis would have equation

$$\arg\left(\frac{z-2}{z+5}\right) = -\frac{\pi}{6}$$

as $\theta - \phi$ would be negative in this picture.



Transformations of the Complex Plane

Always start from the z-plane and transform to the w-plane, $z = x + iy$ and $w = u + iv$.

Example 1: Find the image of the circle $|z - 5| = 3$ under the transformation $w = \frac{1}{z-2}$.

Solution: *First* rearrange to find z

$w = \frac{1}{z-2} \Rightarrow z - 2 = \frac{1}{w} \qquad \Rightarrow \qquad z = \frac{1}{w} + 2$

Second substitute in equation of circle

$\Rightarrow \qquad \left|\frac{1}{w} + 2 - 5\right| = 3 \qquad \Rightarrow \qquad \left|\frac{1-3w}{w}\right| = 3$

$\Rightarrow \qquad |1 - 3w| = 3|w| \qquad \Rightarrow \qquad 3\left|\frac{1}{3} - w\right| = 3|w|$

$\Rightarrow \qquad \left|w - \frac{1}{3}\right| = |w|$

which is the equation of the perpendicular bisector of the line joining 0 to $\frac{1}{3}$,

$\Rightarrow$ the image is the line $u = \frac{1}{6}$

Always consider the 'modulus technique' (above) first;

if this does not work then use the $u + iv$ method shown below.

Example 2: Show that the image of the line $x + 4y = 4$ under the transformation $w = \frac{1}{z-3}$ is a circle, and find its centre and radius.

Solution: *First* rearrange to find $z \Rightarrow z = \frac{1}{w} + 3$

The 'modulus technique' is not suitable here.

$z = x + iy$ and $w = u + iv$

$\Rightarrow \qquad z = \frac{1}{w} + 3 = \frac{1}{u+iv} + 3 = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} + 3$

$\Rightarrow \qquad x + iy = \frac{u-iv}{u^2+v^2} + 3$

Equating real and imaginary parts $x = \frac{u}{u^2+v^2} + 3$ and $y = \frac{-v}{u^2+v^2}$

$\Rightarrow \quad x + 4y = 4$ becomes $\frac{u}{u^2+v^2} + 3 - \frac{4v}{u^2+v^2} = 4$

$\Rightarrow \qquad u^2 - u + v^2 + 4v = 0$

$\Rightarrow \qquad \left(u - \frac{1}{2}\right)^2 + (v + 2)^2 = \frac{17}{4}$

which is a circle with centre $\left(\frac{1}{2}, -2\right)$ and radius $\frac{\sqrt{17}}{2}$.

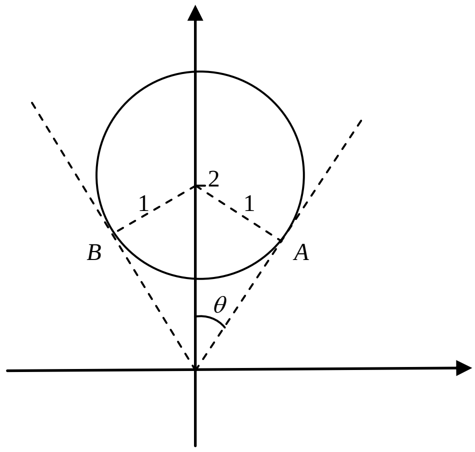
There are many more examples in the book, but these are the two important techniques.

Loci and geometry

It is always important to think of diagrams.

Example: z lies on the circle $|z - 2i| = 1$.
Find the greatest and least values of $\arg z$.

Solution: Draw a picture!



The greatest and least values of $\arg z$ will occur at B and A.

Trigonometry tells us that

$\theta = \frac{\pi}{6}$

and so greatest and least values of

$\arg z$ are $\frac{2\pi}{3}$ and $\frac{\pi}{3}$

4 First Order Differential Equations

Separating the variables, families of curves

Example: Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x(x+1)}, \quad \text{for } x > 0,$$

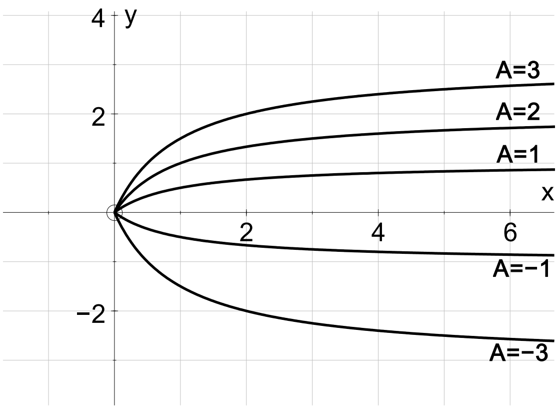
and sketch the family of solution curves.

Solution: $\frac{dy}{dx} = \frac{y}{x(x+1)} \quad \Rightarrow \quad \int \frac{1}{y}\, dy = \int \frac{1}{x(x+1)}\, dx = \int \frac{1}{x} - \frac{1}{x+1}\, dx$

$\Rightarrow \quad \ln y = \ln x - \ln(x+1) + \ln A$

$\Rightarrow \quad y = \frac{Ax}{x+1} = \frac{A(x+1-1)}{x+1} = A\left(1 - \frac{1}{x+1}\right)$

Thus for varying values of A and for $x > 0$, we have



Exact Equations

In an exact the L.H.S. is an exact derivative (really a preparation for Integrating Factors).

Example: Solve $\sin x \frac{dy}{dx} + y\cos x = 3x^2$

Solution: Notice that the L.H.S. is an exact derivative

$\sin x \frac{dy}{dx} + y\cos x = \frac{d}{dx}(y\sin x)$

$\Rightarrow \quad \frac{d}{dx}(y\sin x) = 3x^2$

$\Rightarrow \quad y\sin x = \int 3x^2\, dx = x^3 + c$

$\Rightarrow \quad y = \frac{x^3 + c}{\sin x}$

Integrating Factors

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only.

In this case, multiply both sides by an Integrating Factor, $R = e^{\int P\,dx}$.
The L.H.S. will now be an exact derivative, $\frac{d}{dx}(Ry)$.
Proceed as in the above example.

Example: Solve $x\frac{dy}{dx} + 2y = 1$

Solution: First divide through by x

$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x}$ now in the correct form

Integrating Factor, I.F., is $R = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\Rightarrow \quad x^2\frac{dy}{dx} + 2xy = x$ multiplying by x^2

$\Rightarrow \quad \frac{d}{dx}(x^2 y) = x,$ check that it is an exact derivative

$\Rightarrow \quad x^2 y = \int x\,dx = \frac{x^2}{2} + c$

$\Rightarrow \quad y = \frac{1}{2} + \frac{c}{x^2}$

Using substitutions

Example 1*:* Use the substitution $y = vx$ (where v is a function of x) to solve the equation

$$\frac{dy}{dx} = \frac{3yx^2 + y^3}{x^3 + xy^2}.$$

Solution: $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3yx^2 + y^3}{x^3 + xy^2} \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{3(vx)x^2 + (vx)^3}{x^3 + x(vx)^2} = \frac{3v + v^3}{1+v^2}$

and we can now separate the variables

$\Rightarrow \quad x\frac{dv}{dx} = \frac{3v + v^3}{1+v^2} - v = \frac{3v + v^3 - v - v^3}{1+v^2} = \frac{2v}{1+v^2}$

$\Rightarrow \quad \frac{1+v^2}{2v}\frac{dv}{dx} = \frac{1}{x}$

$\Rightarrow \quad \int \frac{1}{2v} + \frac{v}{2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \quad \frac{1}{2}\ln v + \frac{v^2}{4} = \ln x + c$

But $v = \frac{y}{x}$, $\quad \Rightarrow \quad \frac{1}{2}\ln\frac{y}{x} + \frac{y^2}{4x^2} = \ln x + c$

$\Rightarrow \quad 2x^2 \ln y + y^2 = 6x^2 \ln x + c'x^2$ c' is new arbitrary constant

and I would not like to find y!!!

Example 2: Use the substitution $y = \frac{1}{z}$ to solve the differential equation $\frac{dy}{dx} = y^2 + y\cot x$.

Solution: $y = \frac{1}{z} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{z^2}\frac{dz}{dx}$

$\Rightarrow \quad \frac{-1}{z^2}\frac{dz}{dx} = \frac{1}{z^2} + \frac{1}{z}\cot x$

$\Rightarrow \quad \frac{dz}{dx} + z\cot x = -1$

Integrating factor is $R = e^{\int \cot x \; dx} = e^{\ln(\sin x)} = \sin x$

$\Rightarrow \quad \sin x \frac{dz}{dx} + z\cos x = -\sin x$

$\Rightarrow \quad \frac{d}{dx}(z\sin x) = -\sin x$ check that it is an exact derivative

$\Rightarrow \quad z\sin x = \cos x + c$

$\Rightarrow \quad z = \frac{\cos x + c}{\sin x}$ but $z = \frac{1}{y}$

$\Rightarrow \quad y = \frac{\sin x}{\cos x + c}$

Example 3: Use the substitution $z = x + y$ to solve the differential equation $\frac{dy}{dx} = \cos(x + y)$

Solution: $z = x + y \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \quad \frac{dz}{dx} = 1 + \cos z$

$\Rightarrow \quad \int \frac{1}{1+\cos z}\, dz = \int dx$ separating the variables

$\Rightarrow \quad \int \frac{1}{2}\sec^2\left(\frac{z}{2}\right)\, dz = x + c$ $1 + \cos z = 1 + 2\cos^2\left(\frac{z}{2}\right) - 1 = 2\cos^2\left(\frac{z}{2}\right)$

$\Rightarrow \quad \tan\left(\frac{z}{2}\right) = x + c$

But $z = x + y \quad \Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + c$

5 Second Order Differential Equations

Linear with constant coefficients

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where a, b and c are constants.

(1) when *f*(*x*) = 0

First write down the Auxiliary Equation, A.E

A.E. $am^2 + bm + c = 0$

and solve to find the roots $m = \alpha$ or β

(i) If α and β are both real numbers, and if $\alpha \neq \beta$
then the Complimentary Function, C.F., is
$y = A\,e^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants of integration

(ii) If α and β are both real numbers, and if $\alpha = \beta$
then the Complimentary Function, C.F., is
$y = (A + Bx)\,e^{\alpha x}$, where A and B are arbitrary constants of integration

(iii) If α and β are both complex numbers, and if $\alpha = a + ib,\ \beta = a - ib$
then the Complimentary Function, C.F.,
$y = e^{\alpha x}(A\sin bx + B\cos bx)$,
where A and B are arbitrary constants of integration

Example 1*:* Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

Solution: A.E. is $m^2 + 2m - 3 = 0$
$\Rightarrow$ $(m - 1)(m + 3) = 0$
$\Rightarrow$ $m = 1$ or -3
$\Rightarrow$ $y = Ae^{x} + Be^{-3x}$ when *f*(*x*) = 0, the C.F. is the solution

Example 2*:* Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Solution: A.E. is $m^2 + 6m + 9 = 0$
$\Rightarrow$ $(m + 3)^2 = 0$
$\Rightarrow$ $m = -3$ (and -3) repeated root
$\Rightarrow$ $y = (A + Bx)e^{-3x}$ when *f*(*x*) = 0, the C.F. is the solution

Example 3: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Solution: A.E. is $m^2 + 4m + 13 = 0$

$\Rightarrow$ $(m + 2)^2 - (3i)^2 = 0$

$\Rightarrow$ $(m + 2 + 3i)(m + 2 - 3i) = 0$

$\Rightarrow$ $m = -2 - 3i$ or $-2 + 3i$

$\Rightarrow$ $y = e^{-2x}(A \sin 3x + B \cos 3x)$ when $f(x) = 0$, the C.F. is the solution

(2) when $f(x) \neq 0$, Particular Integrals

First proceed as in (1) to find the Complimentary Function, then use the rules below to find a Particular Integral, P.I.

Second the General Solution, G.S. , is found by adding the C.F. and the P.I.

$\Rightarrow$ **G.S. = C.F. + P.I.**

Note that it does not matter what P.I. you use, so you might as well find the easiest, which is what these rules do.

(1) $f(x) = e^{kx}$.

Try $y = Ae^{kx}$
unless e^{kx} appears in the C.F., in which case try $y = Cxe^{kx}$
unless xe^{kx} appears in the C.F., in which case try $y = Cx^2e^{kx}$.

(2) $f(x) = \sin kx$ or $f(x) = \cos kx$

Try $y = C \sin kx + D \cos kx$
unless $\sin kx$ or $\cos kx$ appear in the C.F., in which case
try $y = x(C \sin kx + D \cos kx)$

(3) $f(x)$ = a polynomial of degree n.

Try $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$
unless a number, on its own, appears in the C.F., in which case
try $f(x) = x(a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0)$

(4) In general

to find a P.I., try something like $f(x)$, unless this appears in the C.F. (or if there is a problem), then try something like $x f(x)$.

Example 1: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2x$

Solution: A.E. is $m^2 + 6m + 5 = 0$

$\Rightarrow$ $(m + 5)(m + 1) = 0$ $\Rightarrow$ $m = -5$ or -1

$\Rightarrow$ C.F. is $y = Ae^{-5x} + Be^{-x}$

For the P.I., try $y = Cx + D$

$\Rightarrow$ $\frac{dy}{dx} = C$ and $\frac{d^2y}{dx^2} = 0$

Substituting in the differential equation gives

$0 + 6C + 5(Cx + D) = 2x$

$\Rightarrow$ $5C = 2$ comparing coefficients of x

$\Rightarrow$ $C = \frac{2}{5}$

and $6C + 5D = 0$ comparing constant terms

$\Rightarrow$ $D = \frac{-12}{25}$

$\Rightarrow$ P.I. is $y = \frac{2}{5}x - \frac{12}{25}$

$\Rightarrow$ G.S. is $y = Ae^{-5x} + Be^{-x} + \frac{2}{5}x - \frac{12}{25}$

Example 2: Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$

Solution: A.E. is is $m^2 - 6m + 9 = 0$

$\Rightarrow$ $(m - 3)^2 = 0$

$\Rightarrow$ $m = 3$ repeated root

$\Rightarrow$ C.F. is $y = (Ax + B)e^{3x}$

In this case, both e^{3x} and xe^{3x} appear in the C.F.,
so for a P.I. we try $y = Cx^2e^{3x}$

$\Rightarrow$ $\frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x}$

and $\frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$

Substituting in the differential equation gives

$2Ce^{3x} + 12Cxe^{3x} + 9Cx^2e^{3x} - 6(2Cxe^{3x} + 3Cx^2e^{3x}) + 9Cx^2e^{3x} = e^{3x}$

$\Rightarrow$ $2Ce^{3x} = e^{3x}$

$\Rightarrow$ $C = \frac{1}{2}$

$\Rightarrow$ P.I. is $y = \frac{1}{2}x^2e^{3x}$

$\Rightarrow$ G.S. is $y = (Ax + B)e^{3x} + \frac{1}{2}x^2e^{3x}$

Example 3: Solve $\frac{d^2x}{dt^2} - x = 4\cos 2t$
given that $x = 0$ and $\dot{x} = 1$ when $t = 0$.

Solution: A.E. is $m^2 - 1 = 0$

$\Rightarrow$ $m = \pm 1$

$\Rightarrow$ C.F. is $x = Ae^t + Be^{-t}$

For the P.I. try $x = C\sin 2t + D\cos 2t$

$\Rightarrow$ $\dot{x} = 2C\cos 2t - 2D\sin 2t$

and $\ddot{x} = -4C\sin 2t - 4D\cos 2t$

Substituting in the differential equation gives

$(-4C\sin 2t - 4D\cos 2t) - (C\sin 2t + D\cos 2t) = 4\cos 2t$

$\Rightarrow$ $-5C = 0$ comparing coefficients of $\sin 2t$

and $-5D = 4$ comparing coefficients of $\cos 2t$

$\Rightarrow$ $C = 0$ and $D = \frac{-5}{4}$

$\Rightarrow$ P.I. is $x = \frac{-5}{4}\cos 2t$

$\Rightarrow$ G.S. is $x = Ae^t + Be^{-t} - \frac{5}{4}\cos 2t$

$\Rightarrow$ $\dot{x} = Ae^t - Be^{-t} + \frac{5}{2}\sin 2t$

$x = 0$ and when $t = 0$ $\Rightarrow$ $0 = A + B - \frac{5}{4}$

and $\dot{x} = 1$ when $t = 0$ $\Rightarrow$ $1 = A - B$

$\Rightarrow$ $A = \frac{9}{8}$ and $B = \frac{1}{8}$

$\Rightarrow$ solution is $x = \frac{9}{8}e^t + \frac{1}{8}e^{-t} - \frac{5}{4}\cos 2t$

D.E.s of the form $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$

Substitute $x = e^u$

$\Rightarrow \quad \frac{dx}{du} = e^u = x$

and $\frac{dy}{du} = \frac{dx}{du} \times \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{du} = x\frac{dy}{dx}$ result I

But $\frac{d^2y}{du^2} = \frac{d\left(\frac{dy}{du}\right)}{du} = \frac{d\left(\frac{dy}{du}\right)}{dx} \times \frac{dx}{du}$ using the chain rule

$= \frac{d\left(x\frac{dy}{dx}\right)}{dx} \times \frac{dx}{du}$ using result I

$= \left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) \times \frac{dx}{du}$ product rule

$\Rightarrow \quad \frac{d^2y}{du^2} = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ since $\frac{dx}{du} = x$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ using result I

Thus we have $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ and $x\frac{dy}{dx} = \frac{dy}{du}$

substituting these in the original equation leads to a second order D.E. with constant coefficients.

Example: Solve the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = -2x^2$.

Solution: Using the substitution $x = e^u$, and proceeding as above

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ and $x\frac{dy}{dx} = \frac{dy}{du}$

$\Rightarrow \quad \frac{d^2y}{du^2} - \frac{dy}{du} - 3\frac{dy}{du} + 3y = -2e^{2u}$

$\Rightarrow \quad \frac{d^2y}{du^2} - 4\frac{dy}{du} + 3y = -2e^{2u}$

$\Rightarrow$ A.E. is $m^2 - 4m + 3 = 0$

$\Rightarrow \quad (m-3)(m-1) = 0 \quad \Rightarrow \quad m = 3$ or 1

$\Rightarrow$ C.F. is $y = Ae^{3u} + Be^u$

For the P.I. try $y = Ce^{2u}$

$\Rightarrow \quad \frac{dy}{du} = 2Ce^{2u}$ and $\frac{d^2y}{du^2} = 4Ce^{2u}$

$\Rightarrow \quad 4Ce^{2u} - 8Ce^{2u} + 3Ce^{2u} = -2e^{2u}$

$\Rightarrow \quad C = 2$

$\Rightarrow$ G.S. is $y = Ae^{3u} + Be^u + 2e^{2u}$

But $x = e^u \quad \Rightarrow$ G.S. is $y = Ax^3 + Bx + 2x^2$

6 Maclaurin and Taylor Series

1) Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

2) Taylor series

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^n}{n!}f^n(a) + \cdots$$

3) Taylor series – as a power series in $(x - a)$

replacing x by $(x - a)$ in 2) we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

4) Solving differential equations using Taylor series

(a) If we are given the value of y when $x = 0$, then we use the Maclaurin series with

$f(0) = y_0$ — the value of y when $x = 0$

$f'(0) = \left(\frac{dy}{dx}\right)_0$ — the value of $\frac{dy}{dx}$ when $x = 0$

etc. to give

$$f(x) = y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \cdots + \frac{x^n}{n!}\left(\frac{d^ny}{dx^n}\right)_0 + \cdots$$

(b) If we are given the value of y when $x = a$, then we use the Taylor power series with

$f(a) = y_a$ — the value of y when $x = a$

$f'(a) = \left(\frac{dy}{dx}\right)_a$ — the value of $\frac{dy}{dx}$ when $x = a$

etc. to give

$$y = y_a + (x-a)\left(\frac{dy}{dx}\right)_a + \frac{(x-a)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_a + \frac{(x-a)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_a + \cdots$$

NOTE THAT 4 (a) and 4 (b) are not in the formula book, but can easily be found using the results in 1) and 3).

Standard series

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ converges for all real x

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$ converges for all real x

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots$ converges for all real x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$ converges for $-1 < x \le 1$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$ converges for $-1 < x < 1$

Example 1: Find the Maclaurin series for $f(x) = \tan x$, up to and including the term in x^3

Solution: $f(x) = \tan x \quad \Rightarrow \quad f(0) = 0$

$\Rightarrow \quad f'(x) = \sec^2 x \quad \Rightarrow \quad f'(0) = 1$

$\Rightarrow \quad f''(x) = 2\sec^2 x \tan x \quad \Rightarrow \quad f''(0) = 0$

$\Rightarrow \quad f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \Rightarrow \quad f^{iv}(0) = 2$

and $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \ldots$

$\Rightarrow \quad \tan x \cong 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2$ up to the term in x^3

$\Rightarrow \quad \tan x \cong x + \frac{x^3}{3}$

Example 2: Using the Maclaurin series for e^x to find an expansion of e^{x+x^2}, up to and including the term in x^3.

Solution: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\Rightarrow \quad e^{x+x^2} \cong 1 + (x + x^2) + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!}$ up to the term in x^3

$\cong 1 + x + x^2 + \frac{x^2 + 2x^3 + \cdots}{2!} + \frac{x^3 + \cdots}{3!}$ up to the term in x^3

$\Rightarrow \quad e^{x+x^2} \cong 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$ up to the term in x^3

Example 3: Find a Taylor series for $\cot\left(x+\frac{\pi}{4}\right)$, up to and including the term in x^2.

Solution: $f(x) = \cot x$ and we are looking for

$$f\left(x+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)$$

$f(x) = \cot x \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = 1$

$\Rightarrow \quad f'(x) = -\text{cosec}^2\, x \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -2$

$\Rightarrow \quad f''(x) = 2\text{cosec}^2\, x \cot x \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = 4$

$\Rightarrow \quad \cot\left(x+\frac{\pi}{4}\right) \cong 1 - 2x + \frac{x^2}{2!} \times 4$ up to the term in x^2

$\Rightarrow \quad \cot\left(x+\frac{\pi}{4}\right) \cong 1 - 2x + 2x^2$ up to the term in x^2

Example 4: Use a Taylor series to solve the differential equation,

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0 \qquad \text{equation I}$$

up to and including the term in x^3, given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

In this case we shall use

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$\Leftrightarrow \quad y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0.$$

We already know that $y_0 = 1$ and $\left(\frac{dy}{dx}\right)_0 = 2$ values when $x = 0$

$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)_0 = \left(-\frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 1\right)_0 = -5$ values when $x = 0$

Differentiating $\quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$

$\Rightarrow \quad y\frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -5$ values when $x = 0$

$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)_0 + 2 \times (^-5) + 2 \times 2 \times (^-5) + 2 = 0$

$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)_0 = 28$

$\Rightarrow \quad$ solution is $\quad y \cong 1 + 2x + \frac{x^2}{2!} \times (^-5) + \frac{x^3}{3!} \times 28$

$\Rightarrow \quad y \cong 1 + 2x - \frac{5}{2}x^2 + \frac{14}{3}x^3$

Series expansions of compound functions

Example: Find a polynomial expansion for

$\frac{\cos 2x}{1-3x}$, up to and including the term in x^3.

Solution: Using the standard series

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots$ up to and including the term in x^3

and $(1-3x)^{-1} = 1 + 3x + \frac{-1\times-2}{2!}(-3x)^2 + \frac{-1\times-2\times-3}{3!}(-3x)^3$

$= 1 + 3x + 9x^2 + 27x^3$ up to and including the term in x^3

$\Rightarrow \frac{\cos 2x}{1-3x} = \left(1 - \frac{(2x)^2}{2!}\right)(1 + 3x + 9x^2 + 27x^3)$

$= 1 + 3x + 9x^2 + 27x^3 - 2x^2 - 6x^3$ up to and including the term in x^3

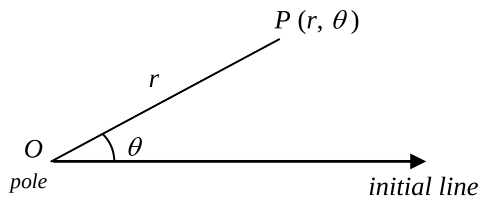
$\Rightarrow \frac{\cos 2x}{1-3x} = 1 + 3x + 7x^2 + 21x^3$ up to and including the term in x^3

7 Polar Coordinates

The polar coordinates of P are $(r,\ \theta)$

$r = OP$, the distance from the origin or *pole*,

and θ is the angle made anti-clockwise with the initial line.



In the Edexcel syllabus *r* is always taken as positive

(But in most books r can be negative, thus $\left(-4,\ \frac{\pi}{2}\right)$ is the same point as $\left(4,\ \frac{3\pi}{2}\right)$)

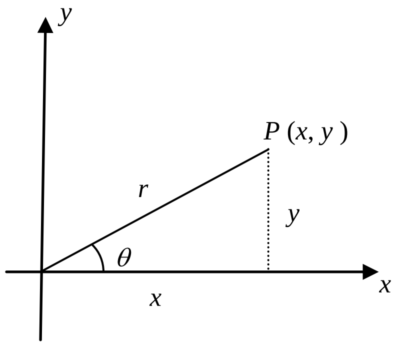
Polar and Cartesian coordinates

From the diagram

$r = \sqrt{x^2 + y^2}$

and $\tan\theta = \frac{y}{x}$ (use sketch to find θ).

$x = r\cos\theta$ and $y = r\sin\theta$.



Sketching curves

In practice, if you are asked to sketch a curve, it will probably be best to plot a few points. The important values of θ are those for which $r = 0$.

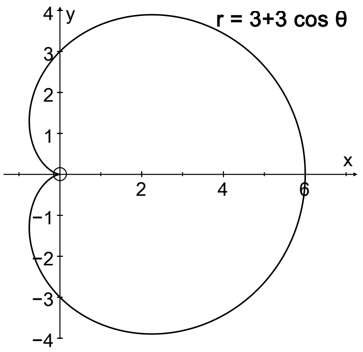
The sketches in these notes will show when r is negative by plotting a dotted line; these sections should be ignored as far as Edexcel A-level is concerned.

Some common curves

$\boldsymbol{r = a + b\cos\theta}$

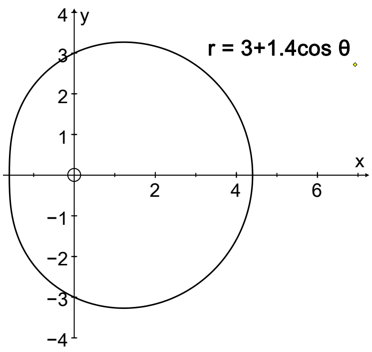
Cardiod

$a = b$



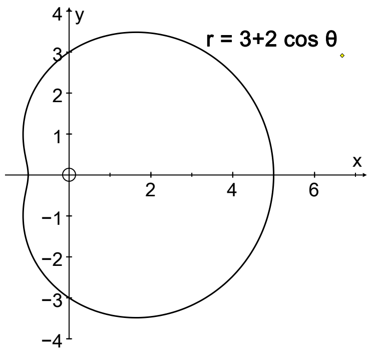
Limacon without dimple

$a \geq 2b$,



Limacon with a dimple

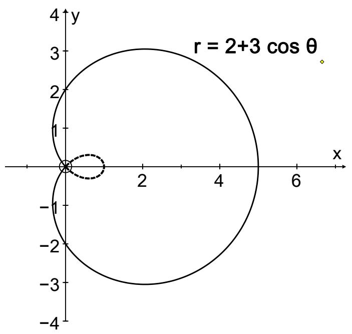
$b \leq a < 2b$



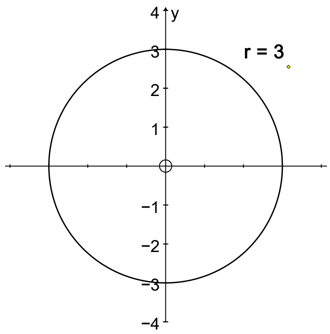
Limacon with a loop

$a < b$

r negative in the loop

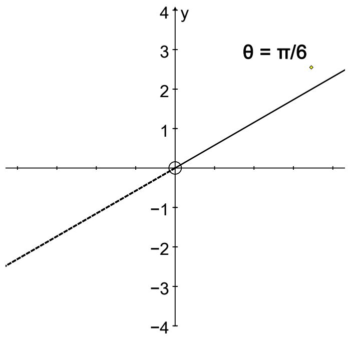


Circle

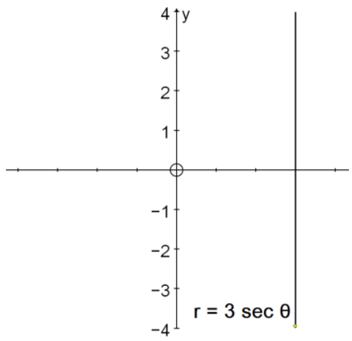


Line

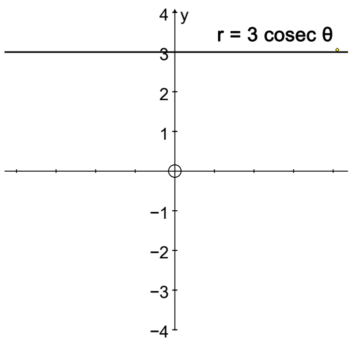
r negative in bottom half



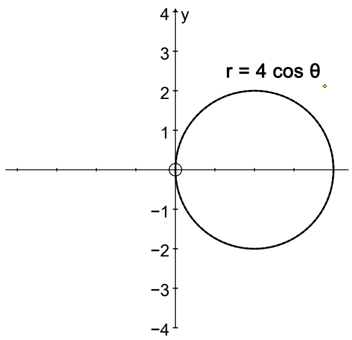
Line



Line



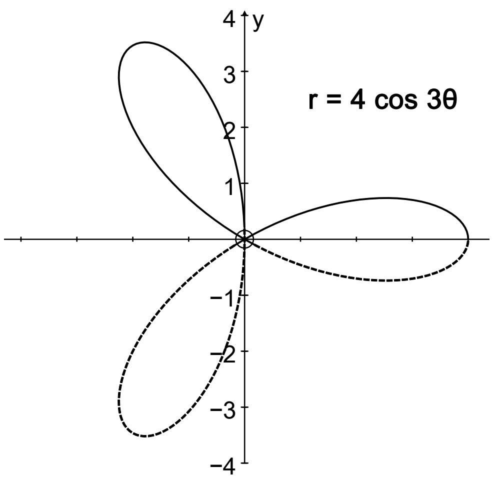
Circle



Rose Curves

$\boldsymbol{r = 4 \cos 3\theta}$

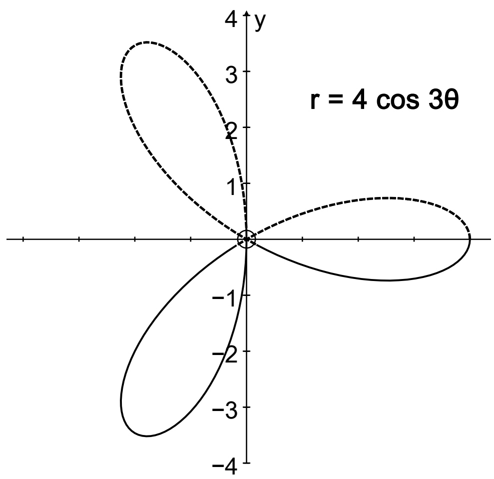
$\boldsymbol{0 \le \theta \le \pi}$



below *x*-axis, *r* negative

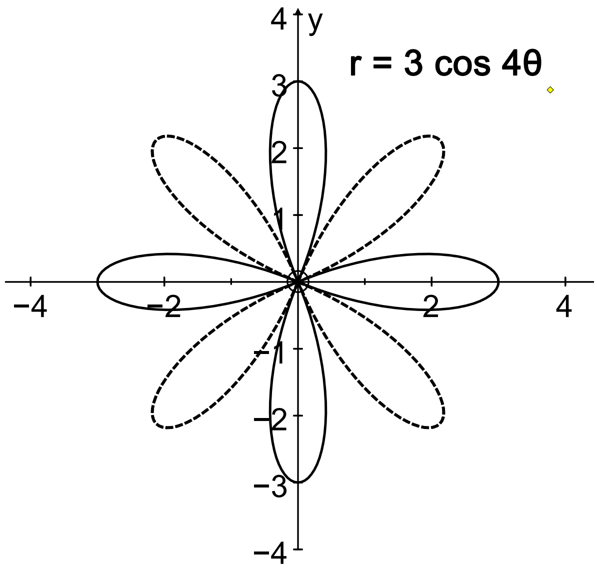
$\boldsymbol{r = 4 \cos 3\theta}$

$\boldsymbol{\pi \le \theta \le 2\pi}$



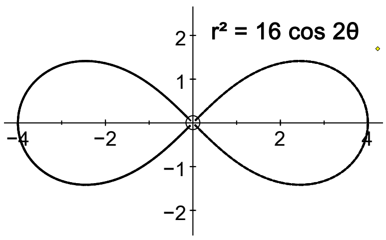
above *x*-axis, *r* negative

$r = 3 \cos 4\theta$

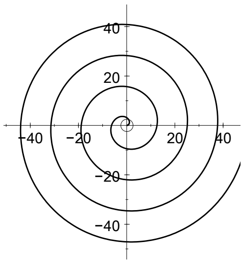


Thus the rose curve $r = a \cos\theta$ always has ***n* petals**, when **only the positive values of *r* are taken.**

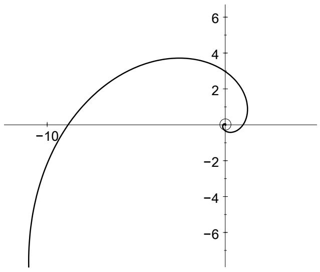
Leminiscate of Bernoulli



Spiral $r = 2\theta$

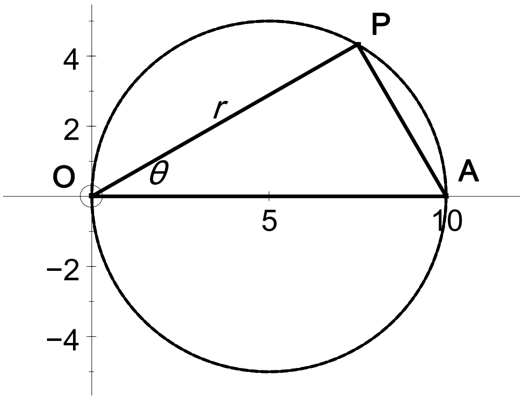


Spiral $r = e^{\theta}$



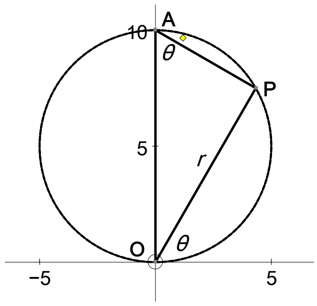
Circle $r = 10 \cos \theta$

Notice that in the circle on OA as diameter, the angle P is 90° (angle in a semi-circle) and trigonometry gives us that $r = 10 \cos \theta$.



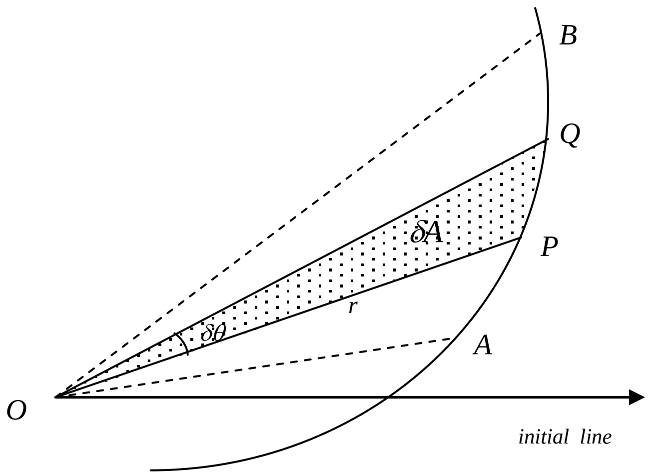
Circle $r = 10 \sin \theta$

In the same way $r = 10 \sin \theta$ gives a circle on the y-axis.



Areas using polar coordinates

Remember: area of a sector is $\frac{1}{2}r^2\theta$



Area of OPQ = $\delta A \approx \frac{1}{2}r^2\delta\theta$

$\Rightarrow$ Area $OAB \approx \sum\left(\frac{1}{2}r^2\delta\theta\right)$

as $\delta\theta \to 0$

$\Rightarrow$ Area $OAB = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta$

Example: Find the area between the curve $r = 1 + \tan\theta$ and the half lines $\theta = 0$ and $\theta = \frac{\pi}{3}$

Solution: Area $= \int_0^{\pi/3} \frac{1}{2}r^2\,d\theta$

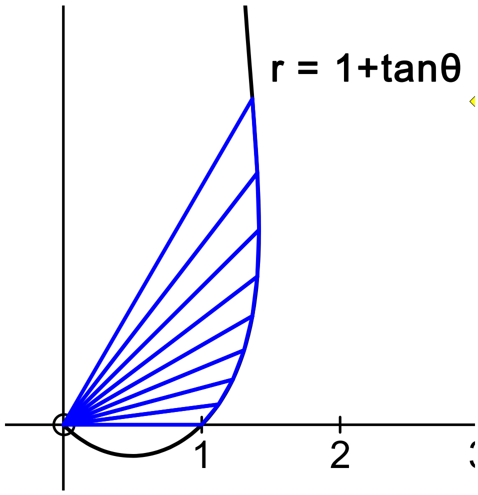
$= \int_0^{\pi/3} \frac{1}{2}(1+\tan\theta)^2\,d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(1+2\tan\theta+\tan^2\theta)\,d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(2\tan\theta+\sec^2\theta)\,d\theta$

$= \frac{1}{2}[2\ln(\sec\theta)+\tan\theta]_0^{\pi/3}$

$= \ln 2 + \frac{\sqrt{3}}{2}$



Tangents parallel and perpendicular to the initial line

$y = r\sin\theta$ and $x = r\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

1) Tangents will be parallel to the initial line ($\theta = 0$), or horizontal, when $\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{d\theta} = 0$

$\Rightarrow \quad \frac{d}{d\theta}(r\sin\theta) = 0$

2) Tangents will be perpendicular to the initial line ($\theta = 0$), or vertical, when $\frac{dy}{dx}$ is infinite

$\Rightarrow \quad \frac{dx}{d\theta} = 0$

$\Rightarrow \quad \frac{d}{d\theta}(r\cos\theta) = 0$

Note that if both $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$, then $\frac{dy}{dx}$ is not defined, and you should look at a sketch to help (or use l'Hôpital's rule).

Example: Find the coordinates of the points on $r = 1 + \cos\theta$ where the tangents are
(*a*) parallel to the initial line,
(*b*) perpendicular to the initial line.

Solution: $r = 1 + \cos\theta$ is shown in the diagram.

(*a*) Tangents parallel to $\theta = 0$ (horizontal)

$\Rightarrow \quad \frac{dy}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(r\sin\theta) = 0$

$\Rightarrow \quad \frac{d}{d\theta}((1+\cos\theta)\sin\theta) = 0 \qquad \Rightarrow \quad \frac{d}{d\theta}(\sin\theta + \sin\theta\cos\theta) = 0$

$\Rightarrow \quad \cos\theta - \sin^2\theta + \cos^2\theta = 0 \qquad \Rightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0 \qquad \Rightarrow \quad \cos\theta = \frac{1}{2} \text{ or } -1$

$\Rightarrow \quad \theta = \pm\frac{\pi}{3} \text{ or } \pi$

(*b*) Tangents perpendicular to $\theta = 0$ (vertical)

$\Rightarrow \quad \frac{dx}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(r\cos\theta) = 0$

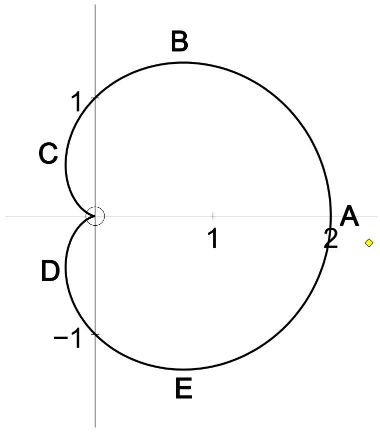
$\Rightarrow \quad \frac{d}{d\theta}((1+\cos\theta)\cos\theta) = 0 \qquad \Rightarrow \quad \frac{d}{d\theta}(\cos\theta + \cos^2\theta) = 0$

$\Rightarrow \quad -\sin\theta - 2\cos\theta\sin\theta = 0 \qquad \Rightarrow \quad \sin\theta\,(1 + 2\cos\theta) = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{2} \text{ or } \sin\theta = 0$

$\Rightarrow \quad \theta = \pm\frac{2\pi}{3} \text{ or } 0, \pi$

From the above we can see that



(a) the tangent is parallel to $\theta = 0$

at B $\left(\theta = \frac{\pi}{3}\right)$, and E $\left(\theta = -\frac{\pi}{3}\right)$,

also at $\theta = \pi$, the origin – see below

(b) the tangent is perpendicular to $\theta = 0$

at A ($\theta = 0$), C $\left(\theta = \frac{2\pi}{3}\right)$ and D $\left(\theta = \frac{-2\pi}{3}\right)$

(c) we also have both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ when $\theta = \pi$!!!

From the graph it looks as if the tangent is parallel to $\theta = 0$ at the origin, $(\theta = \pi)$, and from l'Hôpital's rule it can be shown that this is true.

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