Pure
Further Mathematics 1

Revision Notes

June 2013

Further Pure 1

1 Complex Numbers

Definitions and arithmetical operations

$i = \sqrt{-1}$, so $\sqrt{-16} = 4i$, $\sqrt{-11} = \sqrt{11}\,i$, etc.
These are called *imaginary* numbers

Complex numbers are written as $z = a + bi$, where a and $b \in \mathbb{R}$.
a is the *real part* and b is the *imaginary part*.

+, –, × are defined in the 'sensible' way; division is more complicated.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
$$(a + bi) \times (c + di) = ac + bdi^2 + adi + bci$$
$$= (ac - bd) + (ad + bc)i \qquad \text{since } i^2 = -1$$

So $(3 + 4i) - (7 - 3i) = -4 + 7i$
and $(4 + 3i)(2 - 5i) = 23 - 14i$

Division – this is just rationalising the denominator.

$$\frac{3+4i}{5+2i} = \frac{3+4i}{5+2i} \times \frac{5-2i}{5-2i} \qquad \text{multiply top and bottom by the complex conjugate}$$

$$= \frac{23+14i}{25+4} = \frac{23}{29} + \frac{14}{29}i$$

Complex conjugate

$z = a + bi$
The *complex conjugate* of z is $z^* = \overline{z} = a - bi$

Properties

If $z = a + bi$ and $w = c + di$, then

(i) $\{(a + bi) + (c + di)\}^* = \{(a + c) + (b + d)i\}^*$

$$= \{(a + c) - (b + d)i\}$$

$$= (a - bi) + (c - di)$$

$\Leftrightarrow \quad (z + w)^* = z^* + w^*$

(ii) $\{(a + bi)(c + di)\}^* = \{(ac - bd) + (ad + bc)i\}^*$

$= \{(ac - bd) - (ad + bc)i\}$

$= (a - bi)(c - di)$

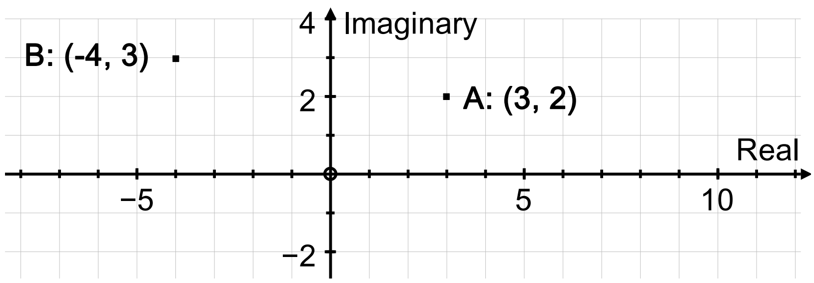
$= (a + bi)^*(c + di)^*$

$\Leftrightarrow \quad (zw)^* = z^* w^*$

Complex number plane, or Argand diagram

We can represent complex numbers as points on the complex number plane:

$3 + 2i$ as the point A (3, 2), and $-4 + 3i$ as the point (–4, 3).



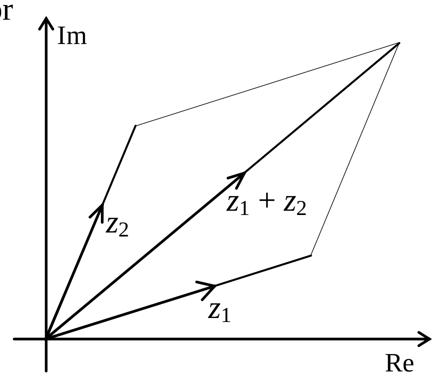
Complex numbers and vectors

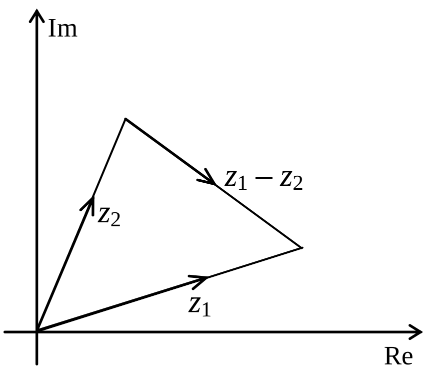
Complex numbers under addition (or subtraction) behave just like vectors under addition (or subtraction). We can show complex numbers on the Argand diagram as **either** points **or** vectors.

(a + bi) + (c + di) = (a + c) + (b + di) $\Leftrightarrow$ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$

(a + bi) – (c + di) = (a – c) + (b – di) $\Leftrightarrow$ $\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$

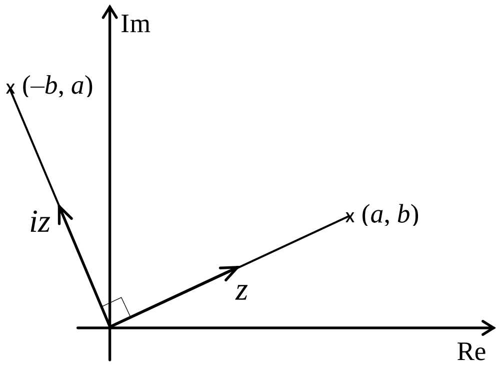
or





Multiplication by *i*

$i(3 + 4i) = -4 + 3i$ – on an Argand diagram this would have the effect of a positive quarter turn about the origin.



In general;

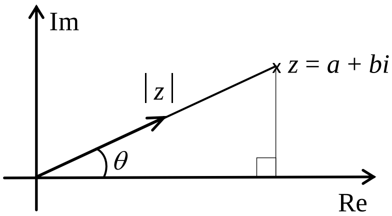
$i(a + bi) = -b + ai$

Modulus of a complex number

This is just like polar co-ordinates.

The modulus of z is $|z|$ and

is the length of the complex number



$$|z| = \sqrt{a^2 + b^2}.$$

$$z\,z^* = (a + bi)(a - bi) = a^2 + b^2$$

$$\Rightarrow \quad z\,z^* = |z|^2.$$

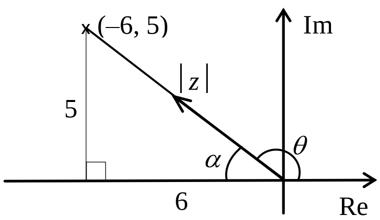
Argument of a complex number

The argument of z is $\arg z$ = the angle made by the complex number with the positive x-axis.
By convention, $-\pi < \arg z \le \pi$.

N.B. **Always draw a diagram when finding arg *z*.**

Example: Find the modulus and argument of $z = -6 + 5i$.

Solution: First sketch a diagram (it is easy to get the argument wrong if you don't).



$$|z| = \sqrt{6^2 + 5^2} = \sqrt{61}$$

and $\tan\alpha = \frac{5}{6} \Rightarrow \alpha = 0{\cdot}694738276$

$\Rightarrow \quad \arg z = \theta = \pi - \alpha = 2.45$ to 3 S.F.

Equality of complex numbers

$a + bi = c + di \quad \Rightarrow \quad a - c = (d - b)i$

$\Rightarrow \quad (a - c)^2 = (d - b)^2 i^2 = -(d - b)^2$ squaring both sides

But $(a - c)^2 \geq 0$ and $-(d - b)^2 \leq 0$

$\Rightarrow \quad (a - c)^2 = -(d - b)^2 = 0$

$\Rightarrow \quad a = c$ and $b = d$

Thus $a + bi = c + di$

$\Rightarrow$ real parts are equal $(a = c)$, and imaginary parts are equal $(b = d)$.

Square roots

Example: Find the square roots of $5 + 12i$, in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution: Let $\sqrt{5 + 12i} = a + bi$

$\Rightarrow \quad 5 + 12i = (a + bi)^2 = a^2 - b^2 + 2abi$

Equating real parts $\Rightarrow \quad a^2 - b^2 = 5, \qquad \mathbf{I}$

equating imaginary parts $\Rightarrow \quad 2ab = 12 \quad \Rightarrow \quad a = \frac{6}{b}$

Substitute in $\mathbf{I}$ $\Rightarrow \quad \left(\frac{6}{b}\right)^2 - b^2 = 5$

$\Rightarrow \quad 36 - b^4 = 5b^2 \quad \Rightarrow \quad b^4 + 5b^2 - 36 = 0$

$\Rightarrow \quad (b^2 - 4)(b^2 + 9) = 0 \quad \Rightarrow \quad b^2 = 4$

$\Rightarrow \quad b = \pm 2$, and $a = \pm 3$

$\Rightarrow \quad \sqrt{5 + 12i} = 3 + 2i$ or $-3 - 2i$.

Roots of equations

(a) **Any** polynomial equation with complex coefficients has a complex solution.

The is *The Fundamental Theorem of Algebra*, and is too difficult to prove at this stage.

Corollary: Any complex polynomial can be factorised into linear factors over the complex numbers.

(b) **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
then the conjugate, $z^* = a - bi$ is also a root.

The proof of this result is in the appendix.

(c) For any polynomial with **real** coefficients, and zeros $a + bi$, $a - bi$,
$(z - (a + bi))(z - (a - bi)) = z^2 - 2az + a^2 - b^2$ will be a quadratic factor in which the coefficients are all **real**.

(d) Using (a), (b), (c), (d) we can see that any polynomial with **real** coefficients can be factorised into a mixture of linear and quadratic factors, all of which have **real** coefficients.

Example: Show that $3 - 2i$ is a root of the equation $z^3 - 8z^2 + 25z - 26 = 0$.
Find the other two roots.

Solution: Put $z = 3 - 2i$ in $z^3 - 8z^2 + 25z - 26$

$= (3 - 2i)^3 - 8(3 - 2i)^2 + 25(3 - 2i) - 26$

$= 27 - 54i + 36i^2 - 8i^3 - 8(9 - 12i + 4i^2) + 75 - 50i - 26$

$= 27 - 36 - 72 + 32 + 75 - 26 + (-54 + 8 + 96 - 50)i$

$= 0 + 0i$

$\Rightarrow$ $3 - 2i$ is a root

$\Rightarrow$ the conjugate, $3 + 2i$, is also a root since all coefficients are **real**

$\Rightarrow$ $(z - (3 + 2i))(z - (3 - 2i)) = z^2 - 6z + 13$ is a factor.

Factorising, by inspection,

$$z^3 - 8z^2 + 25z - 26 = (z^2 - 6z + 13)(z - 2) = 0$$

$\Rightarrow$ roots are $z = 3 \pm 2i$, or 2

2 Numerical solutions of equations

Accuracy of solution

When asked to show that a solution is accurate to n D.P., you must look at the value of $f(x)$ 'half' below and 'half' above, and conclude that

> there is a **change of sign** in the **interval**, and the function is **continuous**, therefore there is a **solution in the interval correct to n D.P.**

Example: Show that $\alpha = 2{\cdot}0946$ is a root of the equation $f(x) = x^3 - 2x - 5 = 0$, accurate to 4 D.P.

Solution:

$f(2.09455) = -0{\cdot}0000165\ldots$, and $f(2.09465) = +0{\cdot}00997$

There is a **change of sign** and f is **continuous**

$\Rightarrow$ there is a **root** in **[2·09455, 2·09465]** $\Rightarrow$ **root is** $\alpha = 2{\cdot}0946$ **to 4 D.P.**

Interval bisection

(i) Find an interval $[a, b]$ which contains the root of an equation $f(x) = 0$.

(ii) $x = \frac{a+b}{2}$ is the mid-point of the interval $[a, b]$

Find $f\left(\frac{a+b}{2}\right)$ to decide whether the root lies in $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$.

(iii) Continue finding the mid-point of each subsequent interval to narrow the interval which contains the root.

Example: (i) Show that there is a root of the equation $f(x) = x^3 - 2x - 7 = 0$ in the interval $[2, 3]$.

(ii) Find an interval of width 0·25 which contains the root.

Solution: (i) $f(2) = 8 - 4 - 7 = -3$, and $f(3) = 27 - 6 - 7 = 14$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

(ii) Mid-point of $[2, 3]$ is $x = 2{\cdot}5$, and $f(2{\cdot}5) = 15{\cdot}625 - 5 - 7 = 3{\cdot}625$

$\Rightarrow$ root in $[2, 2{\cdot}5]$

Mid-point of [2, 2·5] is $x = 2{\cdot}25$,
and $f(2{\cdot}25) = 11{\cdot}390625 - 4{\cdot}5 - 7 = -0{\cdot}109375$

$\Rightarrow$ root in [2·25, 2·5], which is an interval of width 0·25

Linear interpolation

To solve an equation $f(x)$ using linear interpolation.

First, find an interval which contains a root,

second, assume that the curve is a straight line and use similar triangles to find where it crosses the x-axis,
third, repeat the process as often as necessary.

Example: (i) Show that there is a root, α, of the equation
$f(x) = x^3 - 2x - 9 = 0$ in the interval [2, 3].

(ii) Use linear interpolation once to find an approximate value of α.
Give your answer to 3 D.P.

Solution: (i) $f(2) = 8 - 4 - 9 = -5$, and $f(3) = 27 - 6 - 9 = 12$

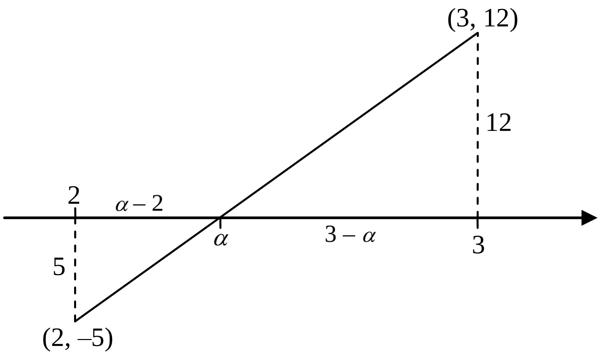
There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in [2, 3].

(ii) From (i), curve passes through $(2, -5)$ and $(3, 12)$, and we assume that the curve is a straight line between these two points.

Let the line cross the x-axis at $(\alpha, 0)$

Using similar triangles

$$\frac{3-\alpha}{\alpha-2} = \frac{12}{5}$$

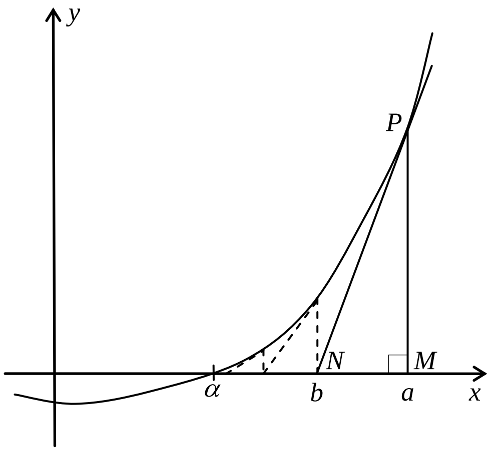


$\Rightarrow$ $15 - 5\alpha = 12\alpha - 24$

$\Rightarrow$ $\alpha = \frac{39}{17} = 2\frac{5}{17}$

$\Rightarrow$ $\alpha = 2{\cdot}294$ to 3 D.P.

Newton-Raphson



Suppose that the equation $f(x) = 0$ has a root at $x = \alpha$, $\Rightarrow f(\alpha) = 0$

To find an approximation for this root, we first find a value $x = a$ near to $x = \alpha$ (decimal search).

In general, the point where the tangent at P, $x = a$, meets the x-axis, $x = b$, will give a better approximation.

At P, $x = a$, the gradient of the tangent is $f'(a)$,

and the gradient of the tangent is also $\frac{PM}{NM}$.

$PM = y = f(a)$ and $NM = a - b$

$\Rightarrow \quad f'(a) = \frac{PM}{NM} = \frac{f(a)}{a-b} \quad \Rightarrow \quad b = a - \frac{f(a)}{f'(a)}$.

Further approximations can be found by repeating the process, which would follow the dotted line converging to the point $(\alpha, 0)$.

This formula can be written as the iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: (i) Show that there is a root, α, of the equation $f(x) = x^3 - 2x - 5 = 0$ in the interval $[2, 3]$.

(ii) Starting with $x_0 = 2$, use the Newton-Raphson formula to find x_1, x_2 and x_3, giving your answers to 3 D.P. where appropriate.

Solution: (i) $f(2) = 8 - 4 - 5 = -1$, and $f(3) = 27 - 6 - 5 = 16$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

(ii) $f(x) = x^3 - 2x - 5 \quad \Rightarrow \quad f'(x) = 3x^2 - 2$

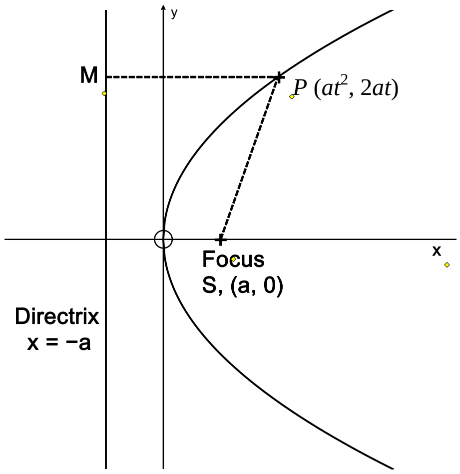
$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8-4-5}{12-2} = 2{\cdot}1$

$\Rightarrow \quad x_2 = 2{\cdot}094568121 = 2{\cdot}095$

$\Rightarrow \quad x_3 = 2{\cdot}094551482 = 2{\cdot}095$

3 Coordinate systems

Parabolas



$y^2 = 4ax$ is the equation of a parabola which passes through the origin and has the x-axis as an axis of symmetry.

Parametric form

$x = at^2$, $y = 2at$ satisfy the equation for all values of t. t is a parameter, and these equations are the parametric equations of the parabola $y^2 = 4ax$.

Focus and directrix

The point S $(a, 0)$ is the *focus*, and

the line $x = -a$ is the *directrix*.

Any point P of the curve is equidistant from the focus and the directrix, $PM = PS$.

Proof: $PM = at^2 - (-a) = at^2 + a$

$PS^2 = (at^2 - a)^2 + (2at)^2 = a^2t^4 - 2a^2t^2 + a^2 + 4a^2t^2$

$= a^2t^4 + 2a^2t^2 + a^2 = (at^2 + a)^2 = PM^2$

$\Rightarrow \quad PM = \text{PS}.$

Gradient

For the parabola $y^2 = 4ax$, with general point P, $(at^2, 2at)$, we can find the gradient in two ways:

1. $y^2 = 4ax$

   $\Rightarrow \quad 2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$

2. At P, $x = at^2$, $y = 2at$

   $\Rightarrow \quad \frac{dy}{dt} = 2a, \quad \frac{dx}{dt} = 2at$

   $\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

Tangents and normals

Example: Find the equations of the tangents to $y^2 = 8x$ at the points where $x = 18$, and show that the tangents meet on the x-axis.

Solution: $x = 18 \Rightarrow y^2 = 8 \times 18 \Rightarrow y = \pm 12$

$2y\frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \pm\frac{1}{3}$ since $y = \pm 12$

$\Rightarrow$ tangents are $y - 12 = \frac{1}{3}(x - 18) \Rightarrow x - 3y + 18 = 0$ at (18, 12)

and $y + 12 = -\frac{1}{3}(x - 18) \Rightarrow x + 3y + 18 = 0.$ at (18, –12)

To find the intersection, add the equations to give

$2x + 36 = 0 \Rightarrow x = -18 \Rightarrow y = 0$

$\Rightarrow$ tangents meet at (–18, 0) on the x-axis.

Example: Find the equation of the normal to the parabola given by $x = 3t^2$, $y = 6t$.

Solution: $x = 3t^2$, $y = 6t \Rightarrow \frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6$,

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{6t} = \frac{1}{t}$

$\Rightarrow$ gradient of the normal is $-t$

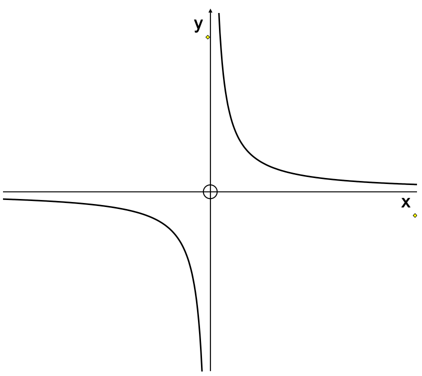
$\Rightarrow$ equation of the normal is $y - 6t = -t(x - 3t^2)$.

Notice that this 'general equation' gives the equation of the normal for any particular value of t:– when $t = -3$ the normal is $y + 18 = 3(x - 27) \Leftrightarrow y = 3x - 99$.

Rectangular hyperbolas

A *rectangular* hyperbola is a hyperbola in which the asymptotes meet at 90°.

$xy = c^2$ is the equation of a rectangular hyperbola in which the x-axis and y-axis are perpendicular asymptotes.



Parametric form

$x = ct$, $y = \frac{c}{t}$ are parametric equations of the hyperbola $xy = c^2$.

Tangents and normals

Example: Find the equation of the tangent to the hyperbola $xy = 36$ at the points where $x = 3$.

Solution: $x = 3 \quad \Rightarrow \quad 3y = 36 \quad \Rightarrow \quad y = 12$

$y = \frac{36}{x} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{36}{x^2} = -4$ when $x = 3$

$\Rightarrow$ tangent is $y - 12 = -4(x - 3) \quad \Rightarrow \quad 4x + y - 24 = 0.$

Example: Find the equation of the normal to the hyperbola given by $x = 3t,\ y = \frac{3}{t}$.

Solution: $x = 3t,\ y = \frac{3}{t} \quad \Rightarrow \quad \frac{dx}{dt} = 3, \quad \frac{dy}{dt} = \frac{-3}{t^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-3}{t^2}}{3} = \frac{-1}{t^2}$

$\Rightarrow$ gradient of the normal is t^2

$\Rightarrow$ equation of the normal is $y - \frac{3}{t} = t^2(x - 3t)$

$\Rightarrow \quad t^3x - y = 3t^4 - 3.$

4 Matrices

You must be able to add, subtract and multiply matrices.

Order of a matrix

An $r \times c$ matrix has r rows and c columns;

the fi**R**st number is the number of **R**ows

the se**C**ond number is the number of **C**olumns.

Identity matrix

The identity matrix is $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that $\boldsymbol{MI} = \boldsymbol{IM} = \boldsymbol{M}$ for *any* matrix $\boldsymbol{M}$.

Determinant and inverse

Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the *determinant* of $\boldsymbol{M}$ is

Det $\boldsymbol{M} = |\boldsymbol{M}| = ad - bc$.

To find the *inverse* of $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Note that $\boldsymbol{M}^{-1}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$

(i) Find the determinant, $ad - bc$. If $ad - bc = 0$, there is no inverse.

(ii) Interchange a and d (the leading diagonal)
Change sign of b and c, (the other diagonal)
Divide all elements by the determinant, $ad - bc$.

$$\Rightarrow \quad \boldsymbol{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Check:

$$\boldsymbol{M}^{-1}\boldsymbol{M} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} da - bc & 0 \\ 0 & -bc + ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

Similarly we could show that $\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$.

Example: $\boldsymbol{M} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $\boldsymbol{MN} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Find $\boldsymbol{N}$.

Solution: Notice that $\boldsymbol{M}^{-1}(\boldsymbol{MN}) = (\boldsymbol{M}^{-1}\boldsymbol{M})\boldsymbol{N} = \boldsymbol{IN} = \boldsymbol{N}$ multiplying on the **left** by $\boldsymbol{M}^{-1}$

But $\boldsymbol{MNM}^{-1} \neq \boldsymbol{IN}$ we can**not** multiply on the **right** by $\boldsymbol{M}^{-1}$

First find $\boldsymbol{M}^{-1}$

Det $\boldsymbol{M} = 4 \times 3 - 2 \times 5 = 2 \quad \Rightarrow \quad \boldsymbol{M}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$

Using $\boldsymbol{M}^{-1}(\boldsymbol{MN}) = \boldsymbol{IN} = \boldsymbol{N}$

$$\Rightarrow \quad \boldsymbol{N} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -7 & 4 \\ 13 & -6 \end{pmatrix} = \begin{pmatrix} -3.5 & 2 \\ 6.5 & -3 \end{pmatrix}.$$

Singular and non-singular matrices

If det $\boldsymbol{A} = 0$, then $\boldsymbol{A}$ is a *singular matrix*, and $\boldsymbol{A}^{-1}$ does not exist.

If det $\boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is a *non-singular matrix*, and $\boldsymbol{A}^{-1}$ exists

Linear Transformations

A matrix can represent a transformation, but the point must be written as a column vector before multiplying by the matrix.

Example: The image of (2, 3) under $\boldsymbol{T} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$ is given by $\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 23 \\ 8 \end{pmatrix}$

$\Rightarrow$ the image of (2, 3) is (23, 8).

Note that the image of (0, 0) is always (0, 0)

$\Leftrightarrow$ the **origin never moves** under a matrix (linear) transformation

Basis vectors

The vectors $\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called *basis* vectors, and are particularly important in describing the geometrical effect of a matrix, and in finding the matrix for a particular geometric transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ c \end{pmatrix}$, the *first* column, and $\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} b \\ d \end{pmatrix}$, the *second* column

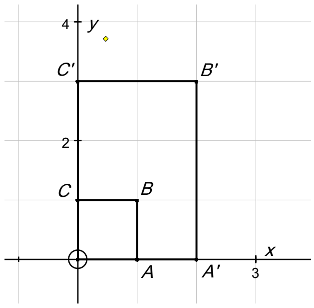
This is a more important result than it seems!

Finding the geometric effect of a matrix transformation

We can easily write down the images of $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$, sketch them and find the geometrical transformation.

Example: Find the geometrical effect of the matrix $\boldsymbol{T} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Solution: Find images of $\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and show on a sketch. Make sure that you letter the points



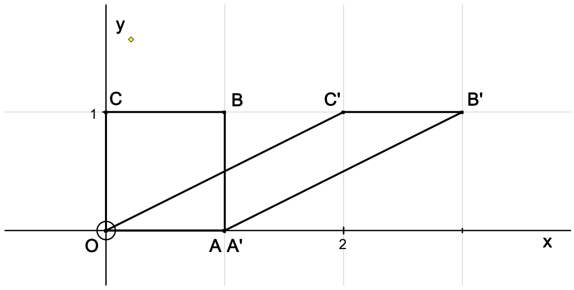
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

From sketch we can see that the transformation is a two-way stretch, of factor 2 parallel to the x-axis and of factor 3 parallel to the y-axis.

Finding the matrix of a given transformation.

Example: Find the matrix for a shear with factor 2 and invariant line the x-axis.

Solution: Each point is moved in the x-direction by a distance of $(2 \times$ its y-coordinate).



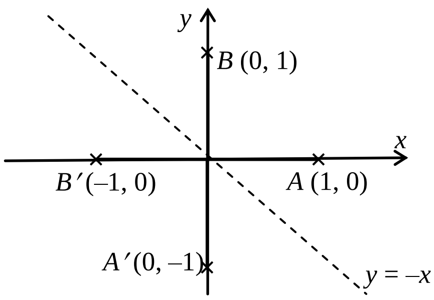
$\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (does not move as it is on the invariant line). This will be the first column of the matrix $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$

$\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This will be the second column of the matrix $\begin{pmatrix} * & 2 \\ * & 1 \end{pmatrix}$

$\Rightarrow$ Matrix of the shear is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Example: Find the matrix for a reflection in the line $y = -x$.

Solution: First find the images of $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$. These will be the two columns of the matrix.



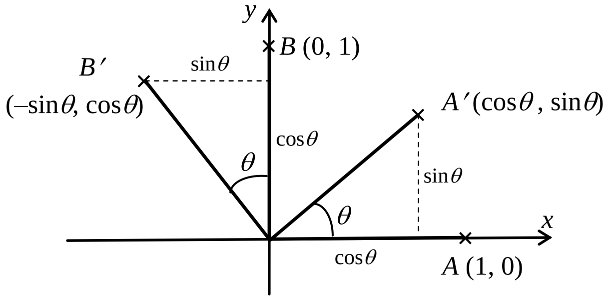
$\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. This will be the first column of the matrix $\begin{pmatrix} 0 & * \\ -1 & * \end{pmatrix}$

$\underline{j} = \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-1\\0\end{pmatrix}$. This will be the second column of the matrix $\begin{pmatrix}* & -1\\ * & 0\end{pmatrix}$

$\Rightarrow$ Matrix of the reflection is $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$.

Rotation matrix

From the diagram we can see that



$\underline{i} = \begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$,

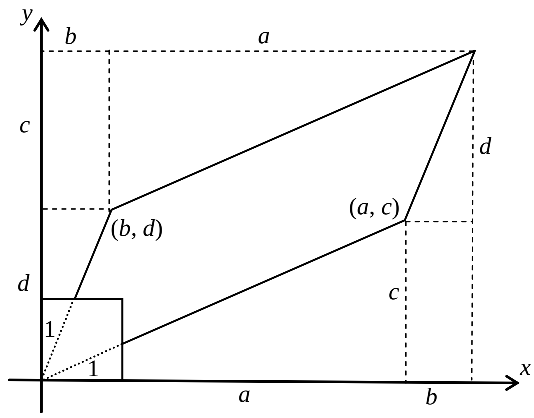
$\underline{j} = \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$

These will be the first and second columns of the matrix

$\Rightarrow$ matrix is $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.

Determinant and area factor

For the matrix $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$



$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$

and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$

$\Rightarrow$ the unit square is mapped on to the parallelogram as shown in the diagram.

The area of the unit square = 1.

The area of the parallelogram = $(a + b)(c + d) - 2 \times (bc + \frac{1}{2}ac + \frac{1}{2}bd)$

$= \quad ac + ad + bc + bd - 2bc - ac - bd$

$= \quad ad - bc \quad = \det A.$

All squares of the grid are mapped onto congruent parallelograms

$\Rightarrow$ area factor of the transformation is $\det A = ad - bc$.

5 Series

You need to know the following sums

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

$$= \left(\tfrac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$$ a fluke, but it helps to remember it

Example: Find $\sum_{r=1}^{n} r(r^2-3)$.

Solution: $\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$

$= \tfrac{1}{4}n^2(n+1)^2 - 3 \times \tfrac{1}{2}n(n+1)$

$= \tfrac{1}{4}n(n+1)\{n(n+1) - 6\}$

$= \tfrac{1}{4}n(n+1)(n+3)(n-2)$

Example: Find $S_n = 2^2 + 4^2 + 6^2 + \ldots + (2n)^2$.

Solution: $S_n = 2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = 2^2(1^2 + 2^2 + 3^2 + \ldots + n^2)$

$= 4 \times \tfrac{1}{6}n(n+1)(2n+1) = \tfrac{2}{3}n(n+1)(2n+1)$.

Example: Find $\sum_{r=5}^{n+2} r^2$

Solution: $\sum_{r=5}^{n+2} r^2 = \sum_{r=1}^{n+2} r^2 - \sum_{r=1}^{4} r^2$ notice that the top limit is 4 **not** 5

$= \tfrac{1}{6}(n+2)(n+2+1)(2(n+2)+1) - \tfrac{1}{6} \times 4 \times 5 \times 9$

$= \tfrac{1}{6}(n+2)(n+3)(2n+5) - 30$.

6 Proof by induction

1. Show that the result/formula is true for $n = 1$ (and sometimes $n = 2$, 3 ..).
   Conclude
   "*therefore the result/formula ………. is true for* $n = 1$".

2. Make induction assumption
   "*Assume that the result/formula ………. is true for* $n = k$".
   Show that the result/formula must then be true for $n = k + 1$
   Conclude
   "*therefore the result/formula ………. is true for* $n = k + 1$".

3. Final conclusion
   "*therefore the result/formula ………… is true for all positive integers, n, by mathematical induction".*

Summation

Example: Use mathematical induction to prove that

$$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

Solution: When $n = 1$, $S_1 = 1^2 = 1$ and $S_1 = \frac{1}{6} \times 1(1+1)(2 \times 1 + 1) = \frac{1}{6} \times 1 \times 2 \times 3 = 1$

$\Rightarrow$ $S_n = \frac{1}{6}n(n+1)(2n+1)$ is true for $n = 1$.

Assume that the formula is true for $n = k$

$\Rightarrow$ $S_k = 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$

$\Rightarrow$ $S_{k+1} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$$= \tfrac{1}{6}(k+1)\{k(2k+1) + 6(k+1)\}$$

$$= \tfrac{1}{6}(k+1)\{2k^2 + 7k + 6\} = \tfrac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \tfrac{1}{6}(k+1)\{(k+1)+1\}\{2(k+1)+1\}$$

$\Rightarrow$ The formula is true for $n = k + 1$

$\Rightarrow$ $S_n = \frac{1}{6}n(n+1)(2n+1)$ is true for all positive integers, n, by mathematical induction.

Recurrence relations

Example: A sequence, 4, 9, 19, 39, … is defined by the recurrence relation

$u_1 = 4$, $u_{n+1} = 2u_n + 1$. Prove that $u_n = 5 \times 2^{n-1} - 1$.

Solution: When $n = 1$, $u_1 = 4$, and $u_1 = 5 \times 2^{1-1} - 1 = 5 - 1 = 4$, $\Rightarrow$ formula true for $n = 1$.

Assume that the formula is true for $n = k$, $\Rightarrow$ $u_k = 5 \times 2^{k-1} - 1$.

From the recurrence relation,

$$u_{k+1} = 2u_k + 1 = 2(5 \times 2^{k-1} - 1) + 1$$

$$\Rightarrow \quad u_{k+1} = 5 \times 2^k - 2 + 1 = 5 \times 2^{(k+1)-1} - 1$$

$\Rightarrow$ the formula is true for $n = k + 1$

$\Rightarrow$ the formula is true for all positive integers, n, by mathematical induction.

Divisibility problems

Considering $f(k + 1) - f(k)$, will often lead to a proof,

but a more reliable way is to consider $f(k + 1) - m \times f(k)$, where m is chosen to eliminate the exponential term.

Example: Prove that $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n.

Solution: When $n = 1$, $f(1) = 5^1 - 4 - 1 = 0$, which is divisible by 16, and so $f(n)$ is divisible by 16 when $n = 1$.

Assume that the result is true for $n = k$, $\Rightarrow$ $f(k) = 5^k - 4k - 1$ is divisible by 16.

Considering $f(k + 1) - 5 \times f(k)$ we will eliminate the 5^k term.

$$\begin{aligned} f(k+1) - 5 \times f(k) &= (5^{k+1} - 4(k+1) - 1) - 5 \times (5^k - 4k - 1) \\ &= 5^{k+1} - 4k - 4 - 1 - 5^{k+1} + 20k + 5 = 16k \end{aligned}$$

$\Rightarrow$ $f(k + 1) = 5 \times \boldsymbol{f(k)} + \mathbf{16}k$

Since $\boldsymbol{f(k)}$ is divisible by 16 (induction assumption), and $\mathbf{16}k$ is divisible by 16, then $f(k + 1)$ must be divisible by 16,

$\Rightarrow$ the result is true for $n = k + 1$

$\Rightarrow$ $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n, by mathematical induction.

Example: Prove that $f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers *n*.

Solution: When $n = 1$, $f(1) = 2^{2+3} + 3^{2-1} = 32 + 3 = 35 = 5 \times 7$, and so the result is true for $n = 1$.

Assume that the result is true for $n = k$

$\Rightarrow$ $f(k) = 2^{2k+3} + 3^{2k-1}$ is divisible by 5

We could consider either

$f(k+1) - 2^2 \times f(k)$, which would eliminate the 2^{2k+3} term **I**

or $f(k+1) - 3^2 \times f(k)$, which would eliminate the 3^{2k-1} term **II**

I $\Rightarrow$ $f(k+1) - 2^2 \times f(k) = 2^{2(k+1)+3} + 3^{2(k+1)-1} - 2^2 \times (2^{2k+3} + 3^{2k-1})$

$= 2^{2k+5} + 3^{2k+1} - 2^{2k+5} - 2^2 \times 3^{2k-1}$

$\Rightarrow$ $f(k+1) - 4 \times f(k) = 9 \times 3^{2k-1} - 4 \times 3^{2k-1} = 5 \times 3^{2k-1}$

$\Rightarrow$ $f(k+1) = 4 \times \boldsymbol{f(k)} - \mathbf{5} \times 3^{2k-1}$

Since $\boldsymbol{f(k)}$ is divisible by 5 (induction assumption), and $\mathbf{5} \times 3^{2k-1}$ is divisible by 5, then $f(k+1)$ must be divisible by 5.

$\Rightarrow$ $f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers, *n*, by mathematical induction.

Powers of matrices

Example: If $M = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, prove that $M^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$ for all positive integers *n*.

Solution: When $n = 1$, $M^1 = \begin{pmatrix} 2^1 & 1-2^1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = M$

$\Rightarrow$ the formula is true for $n = 1$.

Assume the result is true for $n = k$ $\Rightarrow$ $M^k = \begin{pmatrix} 2^k & 1-2^k \\ 0 & 1 \end{pmatrix}$.

$$M^{k+1} = MM^k = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 1-2^k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \times 2^k & 2 - 2 \times 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ $M^{k+1} = \begin{pmatrix} 2^{k+1} & 1-2^{k+1} \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$ The formula is true for $n = k + 1$

$\Rightarrow$ $M^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$ is true for all positive integers, *n*, by mathematical induction.

7 Appendix

Complex roots of a real polynomial equation

Preliminary results:

I $(z_1 + z_2 + z_3 + z_4 + \ldots + z_n)^* = z_1^* + z_2^* + z_3^* + z_4^* + \ldots + z_n^*$,

by repeated application of $(z + w)^* = z^* + w^*$

II $(z^n)^* = (z^*)^n$

$(zw)^* = z^*w^*$

$\Rightarrow (z^n)^* = (z^{n-1}z)^* = (z^{n-1})^*(z)^* = (z^{n-2}z)^*(z)^* = (z^{n-2})^*(z)^*(z)^* \ldots = (z^*)^n$

Theorem: **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
then the conjugate, $z^* = a - bi$ is also a root.

Proof: If $z = a + bi$ is a root of the equation $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_1 z + \alpha_0 = 0$

then $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$

$\Rightarrow (\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0)^* = 0$ since $0^* = 0$

$\Rightarrow (\alpha_n z^n)^* + (\alpha_{n-1} z^{n-1})^* + \ldots + (\alpha_2 z^2)^* + (\alpha_1 z)^* + (\alpha_0)^* = 0$ using **I**

$\Rightarrow \alpha_n^*(z^n)^* + \alpha_{n-1}^*(z^{n-1})^* + \ldots + \alpha_2^*(z^2)^* + \alpha_1^*(z)^* + \alpha_0^* = 0$ since $(zw)^* = z^*w^*$

$\Rightarrow \alpha_n(z^n)^* + \alpha_{n-1}(z^{n-1})^* + \ldots + \alpha_2(z^2)^* + \alpha_1(z)^* + \alpha_0 = 0$ α_i **real** $\Rightarrow \alpha_i^* = \alpha_i$

$\Rightarrow \alpha_n(z^*)^n + \alpha_{n-1}(z^*)^{n-1} + \ldots + \alpha_2(z^*)^2 + \alpha_1(z^*) + \alpha_0 = 0$ using **II**

$\Rightarrow z^* = a - bi$ is also a root of the equation.

Formal definition of a linear transformation

A linear transformation $\boldsymbol{T}$ has the following properties:

(i) $\boldsymbol{T}\begin{pmatrix} kx \\ ky \end{pmatrix} = k\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix}$

(ii) $\boldsymbol{T}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \boldsymbol{T}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

It can be shown that **any** matrix transformation is a linear transformation, and that **any** linear transformation can be represented by a matrix.

Derivative of x^n, for any integer

We can use proof by induction to show that $\frac{d}{dx}(x^n) = nx^{n-1}$, for any integer n.

1) We know that the derivative of x^0 is 0 which equals $0x^{-1}$,

   since $x^0 = 1$, and the derivative of 1 is 0

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = 0$.

2) We know that the derivative of x^1 is 1 which equals $1 \times x^{1-1}$

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = 1$

   Assume that the result is true for $n = k$

   $\Rightarrow \quad \frac{d}{dx}(x^k) = kx^{k-1}$

   $\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \times x^k) = x \times \frac{d}{dx}(x^k) + 1 \times x^k$ product rule

   $\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = x \times kx^{k-1} + x^k = kx^k + x^k = (k+1)x^k$

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = k + 1$

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for all positive integers, n, by mathematical induction.

3) We know that the derivative of x^{-1} is $-x^{-2}$ which equals $-1 \times x^{-1-1}$

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = -1$

   Assume that the result is true for $n = k$

   $\Rightarrow \quad \frac{d}{dx}(x^k) = kx^{k-1}$

   $\Rightarrow \quad \frac{d}{dx}(x^{k-1}) = \frac{d}{dx}\left(\frac{x^k}{x}\right) = \frac{x \times \frac{d}{dx}(x^k) - x^k \times 1}{x^2}$ quotient rule

   $\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = \frac{x \times kx^{k-1} - x^k}{x^2} = \frac{(k-1)x^k}{x^2} = (k-1)x^{k-2} = (k-1)x^{(k-1)-1}$

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = k - 1$

   We are going backwards (**from** $n = k$ **to** $n = k - 1$), and, since we started from $n = -1$,

   $\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for all negative integers, n, by mathematical induction.

   Putting 1), 2) and 3), we have proved that

   $\frac{d}{dx}(x^n) = nx^{n-1}$, for **any** integer n.

8 Index