Pure Core 3

Revision Notes

June 2014

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1 Algebraic fractions

Cancelling common factors

Example: Simplify $\dfrac{x^3 - 2x^2 - 3x}{4x^2 - 36}$.

Solution: First factorise top and bottom fully –

$$\frac{x^3 - 2x^2 - 3x}{4x^2 - 36} = \frac{x(x^2 - 2x - 3)}{4(x^2 - 9)} = \frac{x(x-3)(x+1)}{4(x-3)(x+3)}$$

and now cancel all common factors, in this case $(x - 3)$ to give

Answer = $\dfrac{x(x+1)}{4(x+3)}$.

Multiplying and dividing fractions

This is just like multiplying and dividing fractions with numbers and then cancelling common factors as above.

Example: Simplify $\dfrac{9x^2 - 4}{3x^2 - 2x} \div \dfrac{3x^2 - x - 2}{x^3 + x^2}$

Solution: First turn the second fraction upside down and multiply

$$= \frac{9x^2 - 4}{3x^2 - 2x} \times \frac{x^3 + x^2}{3x^2 - x - 2}$$ factorise fully

$$= \frac{(3x-2)(3x+2)}{x(3x-2)} \times \frac{x^2(x+1)}{(3x+2)(x-1)}$$ cancel all common factors

Answer = $\dfrac{x(x+1)}{x-1}$.

Adding and subtracting fractions

Again this is like adding and subtracting fractions with numbers; **but** finding the Lowest Common Denominator can save a lot of trouble later.

Example: Simplify $\dfrac{3x}{x^2 - 7x + 12} - \dfrac{5}{x^2 - 4x + 3}$.

Solution: First factorise the denominators

$$= \frac{3x}{(x-3)(x-4)} - \frac{5}{(x-3)(x-1)}$$ we see that the L.C.D. is $(x - 3)(x - 4)(x - 1)$

$$= \frac{3x(x-1)}{(x-3)(x-4)(x-1)} - \frac{5(x-4)}{(x-3)(x-1)(x-4)}$$

$$= \frac{3x^2 - 3x - 5x + 20}{(x-1)(x-3)(x-4)} = \frac{3x^2 - 8x + 20}{(x-1)(x-3)(x-4)}$$ which cannot be simplified further.

Equations

Example: Solve $\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{2}$

Solution: First multiply **both sides** by the Lowest Common Denominator

$\dfrac{x}{x+1} - \dfrac{x-1}{x} = \dfrac{1}{2}$ multiply both sides by $2x(x + 1)$

$\Rightarrow \quad x \times 2x - (x - 1) \times 2(x + 1) = x(x + 1)$

$\Rightarrow \quad 2x^2 - 2x^2 + 2 = x^2 + x$

$\Rightarrow \quad x^2 + x - 2 = 0,$

$\Rightarrow \quad (x + 2)(x - 1) = 0$

$\Rightarrow \quad x = -2$ or 1

2 Functions

A function is an expression (often in x) which has **only one** value for each value of x.

Notation

$y = x^2 - 3x + 7$, $f(x) = x^2 - 3x + 7$ and $f : x \rightarrow x^2 - 3x + 7$

are all ways of writing the same function.

Domain, range and graph

The *domain* is the set of values which x can take:

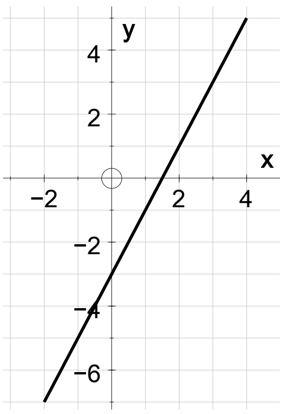
this is sometimes specified in the definition
and sometimes is evident from the function: e.g. $\sqrt{x}$ can only take positive x or zero values.

The *range* is that part of the y–axis which is used.

Example: Find the range of the function

$f : x \rightarrow 2x - 3$ with domain $x \in \Re$: $-2 < x \leq 4$.

Solution: First sketch the graph for values of x between -2 and 4, and we can see that we are only using the y–axis from -7 to 5,



not including $y = -7$ (since $x \neq -2$),

but including $y = 5$ (since x can equal 4)

and so the range is

$y \in \Re$: $-7 < y \leq 5$.

Example: Find the largest possible domain and the range for the function

$f: x \to \sqrt{x-3} + 1$.

Solution: First notice that we cannot have the square root of a negative number and so

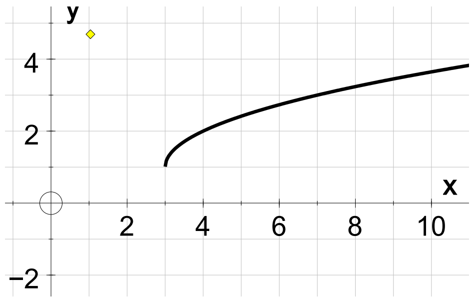
$x - 3$ cannot be negative

$\Rightarrow \quad x - 3 \geq 0$

$\Rightarrow$ largest possible domain is $x \in \Re: \ x \geq 3$.

To find the range we first sketch the graph and we see that the graph will cover all of the *y*-axis from 1 upwards

and so the range is $y \in \Re: \ y \geq 1$.



Example: Find the largest possible domain and the range for the function

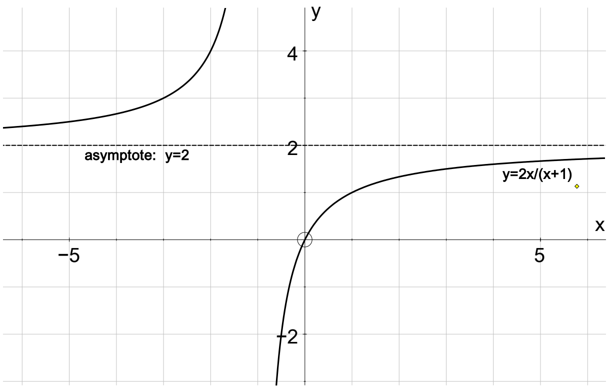
$$f: x \to \frac{2x}{x+1}.$$

Solution: The only problem occurs when the denominator is 0, and so *x* cannot be –1.

Thus the largest domain is $x \in \Re: \ x \neq -1$.

To find the range we sketch the graph and we see that y can take any value except 2,

so the range is $y \in \Re: \ y \neq 2$.



Defining functions

Some mappings can be made into functions by restricting the domain.

Examples:

1) The mapping $x \to \sqrt{x}$ where $x \in \Re$ is not a function as $\sqrt{-9}$ is not defined, but if we restrict the domain to positive or zero real numbers then $f : x \to \sqrt{x}$ where $x \in \Re, x \geq 0$ **is** a function.

2) $x \to \dfrac{1}{x-3}$ where $x \in \Re$ is not a function as the image of $x = 3$ is not defined,

   but $f : x \to \dfrac{1}{x-3}$ where $x \in \Re,\ x \neq 3$ **is** a function.

Composite functions

To find the composite function fg we must do g first.

Example: $f : x \to 3x - 2$ and $g : x \to x^2 + 1$. Find fg and gf.

Solution: Think of f and g as 'rules'

f is [multiply by 3] → [subtract 2]

g is [square] → [add 1]

$\Rightarrow$ fg is [square] → [add 1] → [multiply by 3] → [subtract 2]

giving $(x^2 + 1) \times 3 - 2 = 3x^2 + 1$

$\Rightarrow$ $fg : x \to 3x^2 + 1$ or $fg(x) = 3x^2 + 1$.

gf is [multiply by 3] → [subtract 2] → [square] → [add 1]

giving $(3x - 2)^2 + 1 = 9x^2 - 12x + 5$

$\Rightarrow$ $gf : x \to 9x^2 - 12x + 5$ or $gf(x) = 9x^2 - 12x + 5$.

Note that fg and gf are **not** the same.

Inverse functions and their graphs

The inverse of f is the 'opposite' of f:

thus the inverse of 'multiply by 3' is 'divide by 3'

and the inverse of 'square' is 'square root'.

The inverse of f is written as f^{-1}: note that this does **not** mean '1 over f'.

The *graph* of $y = f^{-1}(x)$ is the reflection in $y = x$ of the graph of $y = f(x)$.

To find the inverse of a function $x \to y$

(i) interchange x and y

(ii) find y in terms of x.

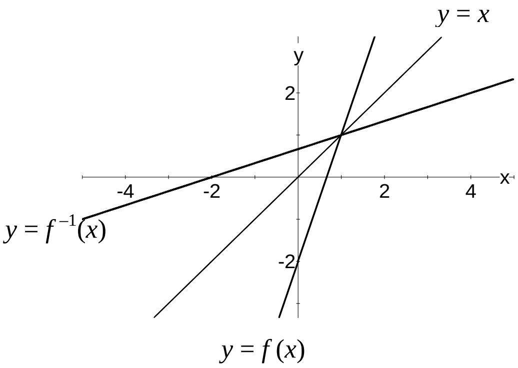
Example: Find the inverse of $f : x \to 3x - 2$.

Solution: We have $x \to y = 3x - 2$

(i) interchanging x and y $\Rightarrow$ $x = 3y - 2$

(ii) solving for y $\Rightarrow$ $y = \dfrac{x+2}{3}$

$\Rightarrow$ $f^{-1} : x \to \dfrac{x+2}{3}$



Example: Find the inverse of $g : x \to \dfrac{x+3}{2x-5}$.

Solution: We have $x \to y = \dfrac{x+3}{2x-5}$

(i) interchanging x and y

$\Rightarrow x = \dfrac{y+3}{2y-5}$

(ii) solving for y

$\Rightarrow x(2y - 5) = y + 3$

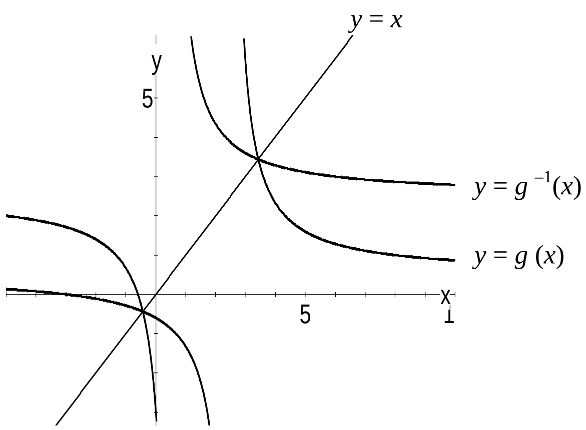
$\Rightarrow 2xy - 5x = y + 3$

$\Rightarrow 2xy - y = 5x + 3$

$\Rightarrow y(2x - 1) = 5x + 3$

$\Rightarrow y = \dfrac{5x+3}{2x-1}$

$\Rightarrow g^{-1} : x \to \dfrac{5x+3}{2x-1}$



Note that $ff^{-1}(x) = f^{-1}f(x) = x$.

Domain and range of inverse functions

Note that the domain of $f(x)$ is the range of $f^{-1}(x)$,

and that the range of $f(x)$ is the domain of $f^{-1}(x)$.

This is because the graph of $y = f^{-1}(x)$ is that of $y = f(x)$ after a reflection in the line $y = x$.

Example: $f(x) = (x-3)^2 + 4, \quad x \in \Re,\ x \le 3.$

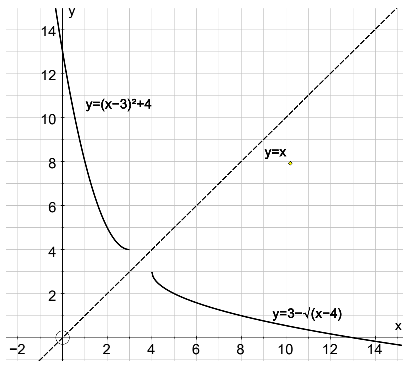
(*a*) Sketch the graph of $y = f(x)$, and state its range.

(*b*) Find the inverse function, $f^{-1}(x)$ and sketch its graph on the same diagram.

Show the line $y = x$ on your diagram.

(*c*) State the domain and range of $f^{-1}(x)$.

Solution:



(*a*) As the domain of f is $x \le 3$, we only have the 'left' part of the parabola.

The range is $f(x) \ge 4,\ f(x) \in \Re$.

(*b*) To find the inverse, swap x and y, then find y.

$$x = (y-3)^2 + 4$$

$$\Rightarrow \quad y - 3 = \pm\sqrt{x-4}\,.$$

From the reflection of $y = f(x)$ in $y = x$, we can see that we want the *negative* sign

$$\Rightarrow \quad y = 3 - \sqrt{x-4}.$$

(*c*) The domain of $f^{-1}(x)$ is $f^{-1}(x) \ge 4$ (the range of $f(x)$).

The range of $f^{-1}(x)$ is $f^{-1}(x) \le 3$ (the domain of $f(x)$).

Modulus functions

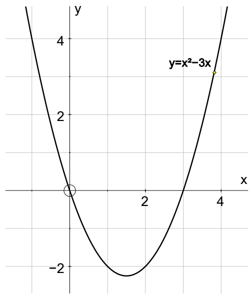
Modulus functions $y = |f(x)|$

$|f(x)|$ is the 'positive value of $f(x)$',

so to sketch the graph of $y = |f(x)|$ first sketch the graph of $y = f(x)$ and then reflect the part(s) below the x–axis to above the x–axis.

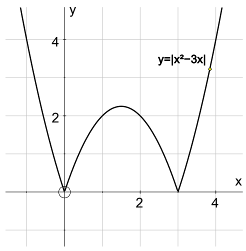
Example: Sketch the graph of $y = |x^2 - 3x|$.

Solution: First sketch the graph of $y = x^2 - 3x$.



Then reflect the portion between $x = 0$ and $x = 3$ in the x–axis

to give



Modulus functions *y* = *f* (|*x*|)

In this case $f(-3) = f(3)$, $f(-5) = f(5)$, $f(-8.7) = f(8.7)$ etc. and so the graph on the left of the y–axis must be the reflection of the graph on the right of the y–axis,

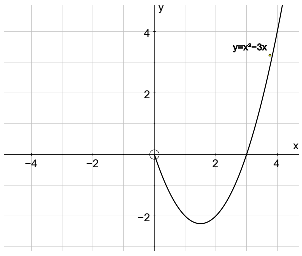
so to sketch the graph, first sketch the graph for **positive values of *x* only**, then reflect the graph sketched in the y–axis.

Example: Sketch the graph of $y = |x|^2 - 3|x|$.

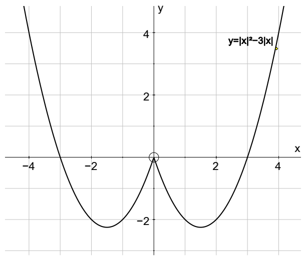
Solution: $f(x) = x^2 - 3x$

$\Rightarrow$ $f(|x|) = |x|^2 - 3|x|$

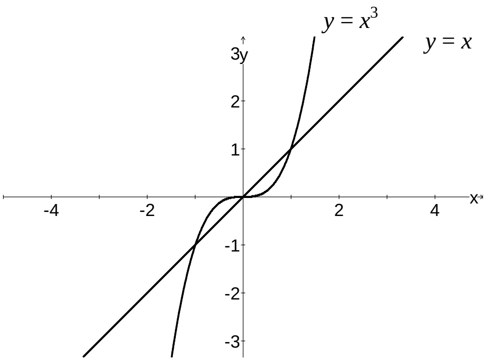
First sketch the graph of $y = x^2 - 3x$ for positive of x only.



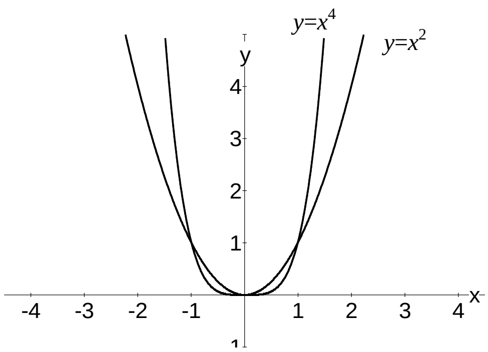
Then reflect your graph in the y–axis to complete the sketch.



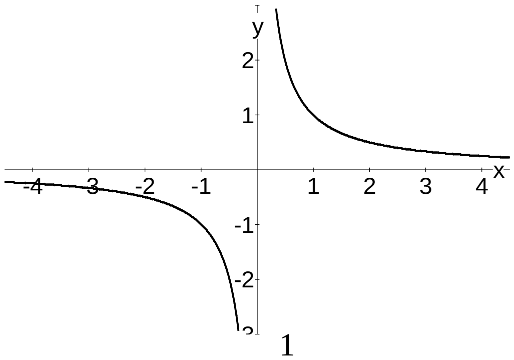
Standard graphs



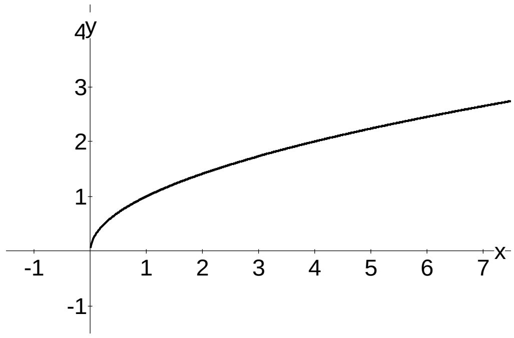
$y = x$ and $y = x^3$



$y = x^2$ and $y = x^4$



$y = \frac{1}{x}$



$y = \sqrt{x}$

Combinations of transformations of graphs

We know the following transformations of graphs:

$y = f(x)$

translated through $\begin{pmatrix} a \\ b \end{pmatrix}$ becomes $y = f(x - a) + b$

stretched factor a in the y-direction becomes $y = a \times f(x)$

stretched factor a in the x-direction becomes $y = f\left(\frac{x}{a}\right)$

reflected in the x-axis becomes $y = -f(x)$

reflected in the y-axis becomes $y = f(-x)$

We can combine these transformations:

Examples:

1) $y = 2f(x - 3)$ is the image of $y = f(x)$ under a stretch in the y-axis of factor 2 followed by a translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, or the translation followed by the stretch.

2) $y = 3x^2 + 6$ is the image of $y = x^2$ under a stretch in the y-axis of factor 3 followed by a translation $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$,

BUT these transformations cannot be done in the reverse order.

To do a translation before a stretch we have to notice that

$3x^2 + 6 = 3(x^2 + 2)$ which is the image of $y = x^2$ under a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

followed by a stretch in the y-axis of factor 3.

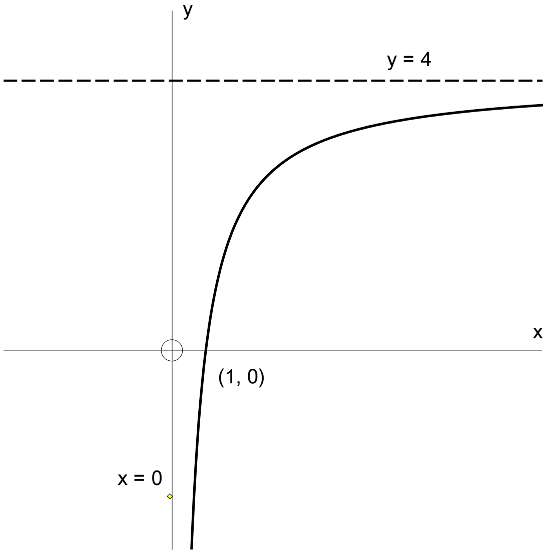
3) $y = -\sin(x + \pi)$ is the image of $y = \sin x$ under a reflection in the x-axis followed by a translation of $\begin{pmatrix} -\pi \\ 0 \end{pmatrix}$, or the translation followed by the reflection.

Sketching curves

When sketching curves, show the coordinates of the intercepts with the axes, and the equations of any asymptotes – show the asymptotes with dotted lines.

Example: Sketch the curve $y = 4 - \frac{2}{x}$, $x > 0$

Solution:



Note that the domain is $x > 0$, so no graph to the left of the y-axis.

$x \neq 0 \Rightarrow$ curve does not meet the x-axis

$y = 0 \Rightarrow x = 1$

Thinking of $y = \frac{-2}{x}$ translated up 4, the horizontal asymptote is $y = 4$.

Do not forget that the y-axis is also an asymptote.

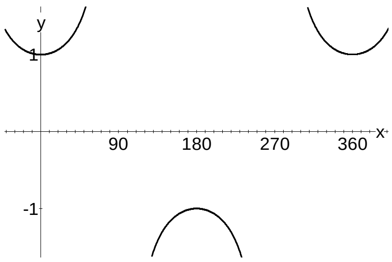
3 Trigonometry

Sec, cosec and cot

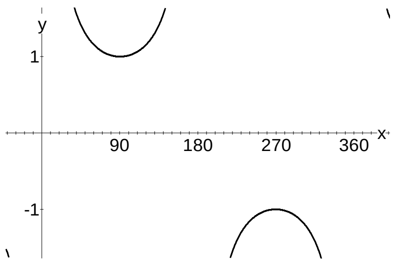
Secant is written sec x = ; y *cosecant* is written cosec x = $\dfrac{1}{\sin x}$

cotangent is written cot x = $\dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

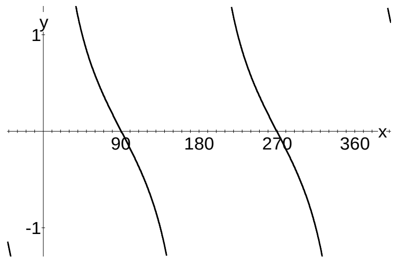
Graphs



y = sec x



y = cosec x



y = cot x

Notice that your calculator does not have sec, cosec and cot buttons so to solve equations involving sec, cosec and cot, change them into equations involving sin, cos and tan and then use your calculator as usual.

Example: Find cosec 35^o.

Solution: cosec 35^o = $\dfrac{1}{\sin 35^o} = \dfrac{1}{0\cdot 53576...} = 1\cdot 743$ to 4 S.F.

Example: Solve sec x = 3·2 for $0 \le x \le 2\pi$

Solution: sec x = 3·2 $\Rightarrow$ $\dfrac{1}{\cos x} = 3\cdot 2 \Rightarrow \Rightarrow$ cos x = $\dfrac{1}{3\cdot 2} = 0\cdot 3125$

$\Rightarrow$ x = 1·25 or 2π – 1·25 = 5·03 radians to 3 S.F.

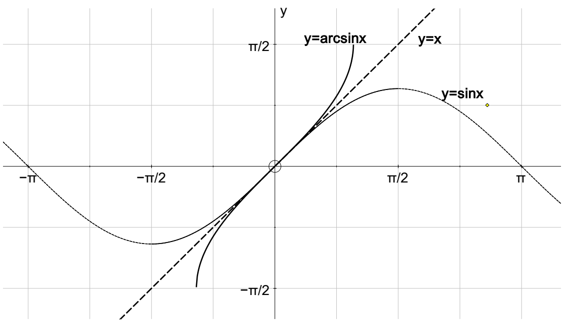
Inverse trigonometrical functions

The inverse of $\sin x$ is written as $\arcsin x$ or $\sin^{-1} x$ and in order that there should only be one value of the function for one value of x we restrict the domain to $-{}^{\pi}/_{2} \leq x \leq {}^{\pi}/_{2}$.

Note that the graph of $y = \arcsin x$ is the reflection of part of the graph of $y = \sin x$ in the line $y = x$.

Similarly for the inverses of $\cos x$ and $\tan x$, as shown below.

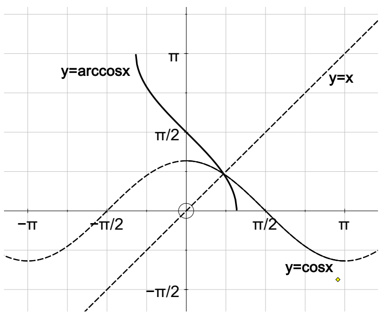
Graphs



$y = \arcsin x$

$-1 \leq x \leq 1$

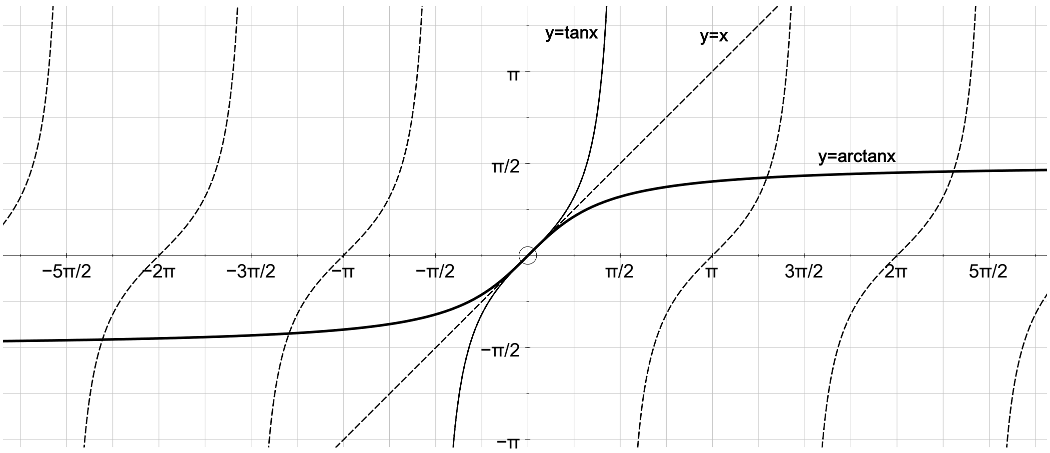
$-{}^{\pi}/_{2} \leq \arcsin x \leq {}^{\pi}/_{2}$



$y = \arccos x$

$-1 \leq x \leq 1$

$0 \leq \arccos x \leq \pi$



$y = \arctan x$

$x \in \Re$

$-{}^{\pi}/_{2} < \arctan x < {}^{\pi}/_{2}$

Trigonometrical identities

You should **learn** these

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\sin 2A = 2\sin A\cos A$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos\theta)$

$\cos^2 \frac{1}{2}\theta = \frac{1}{2}(1 + \cos\theta)$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$2\sin A\cos B = \sin(A + B) + \sin(A - B)$

$2\cos A\sin B = \sin(A + B) - \sin(A - B)$

$2\cos A\cos B = \cos(A + B) + \cos(A - B)$

$-2\sin A\sin B = \cos(A + B) - \cos(A - B)$

The last four formulae, $2\sin A\cos B = \sin(A + B) + \sin(A - B)$ etc., are **not** in the formula booklet, and should be **learnt**.

You should know the proofs of the four formulae $\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$, etc.

Proof of $\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$

We know that $\sin(A + B) = \sin A\cos B + \cos A\sin B$

and $\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\Rightarrow \quad \sin(A + B) - \sin(A - B) = 2\sin A\cos B$

Now put $P = A + B$, and $Q = A - B$

$\Rightarrow \quad P + Q = 2A$, and $P - Q = 2B$, $\quad \Rightarrow \quad A = \frac{P+Q}{2}$, and $B = \frac{P-Q}{2}$

$\Rightarrow \quad \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$.

The other formulae can be proved in a similar way.

Finding exact values

When finding exact values you may **not** use calculators.

Example: Find the exact value of $\cos 15^{\circ}$

Solution: We know the exact values of $\sin 45^{\circ}$, $\cos 45^{\circ}$ and $\sin 30^{\circ}$, $\cos 30^{\circ}$

so we consider $\cos 15 = \cos (45 - 30) = \cos 45 \cos 30 + \sin 45 \sin 30$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Example: Given that A is obtuse and that B is acute, and $\sin A = {}^3/_5$ and $\cos B = {}^5/_{13}$ find the exact value of $\sin(A + B)$.

Solution: We know that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ so we must first find $\cos A$ and $\sin B$.

Using $\sin^2 \theta + \cos^2 \theta = 1$

$\Rightarrow \quad \cos^2 A = 1 - {}^9/_{25} = {}^{16}/_{25}$ and $\sin^2 B = 1 - {}^{25}/_{169} = {}^{144}/_{169}$

$\Rightarrow \quad \cos A = \pm {}^4/_5$ and $\sin B = \pm {}^{12}/_{13}$

But A is obtuse so $\cos A$ is negative, and B is acute so $\sin B$ is positive

$\Rightarrow \quad \cos A = -{}^4/_5$ and $\sin B = {}^{12}/_{13}$

$\Rightarrow \quad \sin(A + B) = {}^3/_5 \times {}^5/_{13} + -{}^4/_5 \times {}^{12}/_{13} = {}^{-33}/_{65}$

Proving identities.

Start with one side, usually the L.H.S., and fiddle with it until it equals the other side.

Do **not** fiddle with both sides at the same time.

Example: Prove that $\dfrac{\cos 2A + 1}{1 - \cos 2A} = \cot^2 A$.

Solution: L.H.S. $= \dfrac{\cos 2A + 1}{1 - \cos 2A} = \dfrac{2\cos^2 A - 1 + 1}{1 - (1 - 2\sin^2 A)} = \dfrac{2\cos^2 A}{2\sin^2 A} = \cot^2 A$. Q.E.D.

Eliminating a variable between two equations

Example: Eliminate θ from the parametric equations $x = \sec\theta - 1$, $y = \tan\theta$.

Solution: We remember that $\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \quad \sec^2\theta - \tan^2\theta = 1.$

$\sec\theta = x + 1$ and $\tan\theta = y$

$\Rightarrow \quad (x+1)^2 - y^2 = 1$

$\Rightarrow \quad y^2 = (x+1)^2 - 1 = x^2 + 2x.$

Solving equations

Here you have to select the 'best' identity to help you solve the equation.

Example: Solve the equation $\sec^2 A = 3 - \tan A$, for $0 \le A \le 360^{o}$.

Solution: We know that $\tan^2 A + 1 = \sec^2 A$

$\Rightarrow \quad \tan^2 A + 1 = 3 - \tan A$

$\Rightarrow \quad \tan^2 A + \tan A - 2 = 0,$ factorising gives

$\Rightarrow \quad (\tan A - 1)(\tan A + 2) = 0$

$\Rightarrow \quad \tan A = 1$ or $\tan A = -2$

$\Rightarrow \quad A = 45, 225,$ or $116{\cdot}6, 296{\cdot}6.$

Example: Solve $\sin 3x - \sin 5x = 0$ for $0^{o} \le x \le 90^{o}$.

Solution: Using the formula $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\Rightarrow \quad 2\cos 4x \sin(-x) = 0 \quad \Rightarrow \quad \cos 4x \sin x = 0$

$\Rightarrow \quad \cos 4x = 0,$ or $\sin x = 0$

$\Rightarrow \quad 4x = 90, 270, (450), \ldots$ or $x = 0, (180), \ldots$

$\Rightarrow \quad x = 0^{o}, 22{\cdot}5^{o}$ or $67{\cdot}5^{o}$.

R cos(x + α)

An alternative way of writing $a \cos x \pm b \sin x$ using one of the formulae listed below

(1) $R \cos(x + \alpha) = R \cos x \cos \alpha - R \sin x \sin \alpha$

(2) $R \cos(x - \alpha) = R \cos x \cos \alpha + R \sin x \sin \alpha$

(3) $R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$

(4) $R \sin(x - \alpha) = R \sin x \cos \alpha - R \cos x \sin \alpha$

To keep R positive and α acute, we select the formula with corresponding + and – signs.
The technique is the same which ever formula we choose.

Example: Solve the equation $12 \sin x - 5 \cos x = 6$ for $0^o \leq x \leq 360^0$.

Solution: First re–write in the above form:

notice that the $\sin x$ is +ve and the $\cos x$ is –ve so we need formula (4).

$R \sin(x - \alpha) = R \sin x \cos \alpha - R \cos x \sin \alpha = 12 \sin x - 5 \cos x$

Equating coefficients of $\sin x$, $\Rightarrow R \cos \alpha = 12$ **I**

Equating coefficients of $\cos x$, $\Rightarrow R \sin \alpha = 5$ **II**

Squaring and adding **I** and **II** $\Rightarrow R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 12^2 + 5^2$

$\Rightarrow R^2 (\cos^2 \alpha + \sin^2 \alpha) = 144 + 25$ but $\cos^2 \alpha + \sin^2 \alpha = 1$

$\Rightarrow R^2 = 169 \Rightarrow R = \pm 13$

But choosing the correct formula means that R is positive $\Rightarrow R = +13$

Substitute in **I** $\Rightarrow \cos \alpha = {}^{12}/_{13}$

$\Rightarrow \alpha = 22{\cdot}620\ldots^o$ or $337{\cdot}379\ldots^o$ or ……

and choosing the correct formula means that α is acute

$\Rightarrow \alpha = 22{\cdot}620\ldots^o$.

$\Rightarrow 12 \sin x - 5 \cos x = 13 \sin(x - 22{\cdot}620\ldots)$

To solve $12 \sin x - 5 \cos x = 6$

$\Rightarrow 12 \sin x - 5 \cos x = 13 \sin(x - 22{\cdot}620\ldots) = 6$

$\Rightarrow \sin(x - 22{\cdot}620\ldots) = {}^{6}/_{13}$

$\Rightarrow x - 22{\cdot}620\ldots = 27{\cdot}486\ldots$ or $180 - 27{\cdot}486\ldots = 152{\cdot}514\ldots$

$\Rightarrow x = 50{\cdot}1^o$ or $175{\cdot}1^o$.

Example: Find the maximum value of $12 \sin x - 5 \cos x$ and the smallest positive value of x for which it occurs.

Solution: From the above example $12 \sin x - 5 \cos x = 13 \sin(x - 22.6)$.

The maximum value of sin(*anything*) is 1 and occurs when the angle is 90, 450, 810 etc. i.e. $90 + 360n$

$\Rightarrow$ the max value of $13 \sin(x - 22{\cdot}6)$ is 13

when $x - 22{\cdot}6 = 90 + 360n \Rightarrow x = 112{\cdot}6 + 360n^{o}$,

$\Rightarrow$ smallest positive value of x is $112{\cdot}6^{o}$.

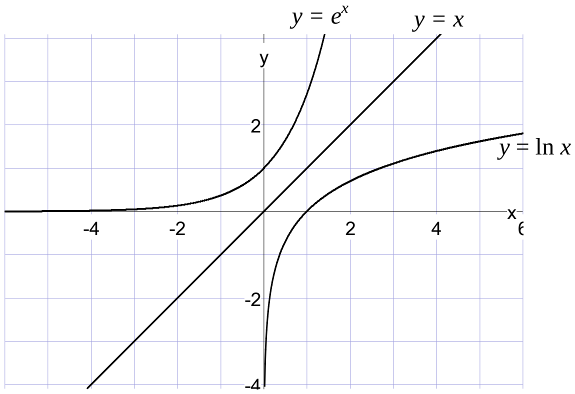
4 Exponentials and logarithms

Natural logarithms

Definition and graph

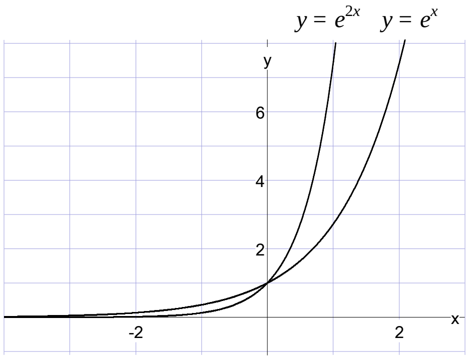
$e \approx 2{\cdot}7183$ and logs to base e are called *natural logarithms*.

$\log_e x$ is usually written $\ln x$.



Note that $y = e^x$ and $y = \ln x$ are inverse functions and that the graph of one is the reflection of the other in the line $y = x$.

Graph of $y = e^{(ax+b)} + c$.



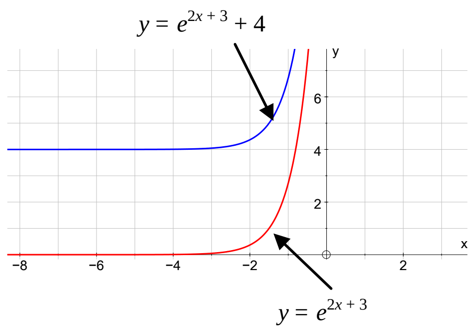
The graph of $y = e^{2x}$ is the graph of $y = e^x$ stretched by a factor of $^1/_2$ in the direction of the x-axis.

$y = e^{2x}$ is above $y = e^x$ for $x > 0$, and below for $x < 0$.

The graph of $y = e^{(2x+3)}$ is the graph of $y = e^{2x}$ translated through $\begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}$ since

$2x + 3 = 2(x - \frac{-3}{2})$

and the graph of $y = e^{(2x+3)} + 4$ is that of $y = e^{(2x+3)}$ translated through $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$



Equations of the form $e^{ax+b} = p$

Example: Solve $e^{2x+3} = 5$.

Solution: Take the natural logarithm of each side, remember that $\ln x$ is the inverse of e^x

$\Rightarrow$ $\ln(e^{2x+3}) = \ln 5$ $\Rightarrow$ $2x + 3 = \ln 5$

$\Rightarrow$ $x = \dfrac{\ln 5 - 3}{2} = -0{\cdot}695$ to 3 S.F.

Example: Solve $\ln(3x - 5) = 4$.

Solution: Raise both sides to the power of e, .

$\Rightarrow$ $e^{\ln(3x-5)} = e^4$ $\Rightarrow$ $(3x - 5) = e^4$ remember that e^x is the inverse of $\ln x$

$\Rightarrow$ $x = \dfrac{e^4 + 5}{3} = 19{\cdot}9$ to 3 S.F.

5 Differentiation

Chain rule

If y is a composite function like $y = (5x^2 - 7)^9$

think of y as $y = u^9$, where $u = 5x^2 - 7$

then the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 9u^8 \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 9(5x^2 - 7)^8 \times (10x) = 90x(5x^2 - 7)^8 .$$

The rule is very simple, just differentiate the function of u and multiply by $\frac{du}{dx}$.

Example: $y = \sqrt{(x^3 - 2x)}$. Find $\frac{dy}{dx}$.

Solution: $y = \sqrt{(x^3 - 2x)} = (x^3 - 2x)^{1/2}$. Put $u = x^3 - 2x$

$$\Rightarrow \quad y = u^{1/2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \tfrac{1}{2}u^{-1/2} \times \frac{du}{dx} = \tfrac{1}{2}(x^3 - 2x)^{-1/2} \times (3x^2 - 2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - 2}{2(x^3 - 2x)^{1/2}} .$$

Product rule

If y is the product of two functions, u and v, then

$$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} .$$

Example: Differentiate $y = x^2 \times \sqrt{(x - 5)}$.

Solution: $y = x^2 \times \sqrt{(x - 5)} = x^2 \times (x - 5)^{1/2}$

so put $u = x^2$ and $v = (x - 5)^{1/2}$

$$\Rightarrow \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \times \tfrac{1}{2}(x - 5)^{-1/2} + (x - 5)^{1/2} \times 2x$$

$$= \frac{x^2}{2\sqrt{x - 5}} + 2x\sqrt{x - 5} .$$

Quotient rule

If y is the quotient of two functions, u and v, then

$$y = \frac{u}{v} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Example: Differentiate $y = \dfrac{2x-3}{x^2+5x}$

Solution: $y = \dfrac{2x-3}{x^2+5x}$, so put $u = 2x - 3$ and $v = x^2 + 5x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(x^2+5x)\times 2 \;-\; (2x-3)\times(2x+5)}{(x^2+5x)^2}$$

$$= \frac{2x^2+10x \;-\; (4x^2+4x-15)}{(x^2+5x)^2} = \frac{-2x^2+6x+15}{(x^2+5x)^2}.$$

Example: If $y = \frac{3x-2}{\sqrt{x-1}}$, find $\frac{dy}{dx}$, expressing your answer as a single algebraic fraction in its simplest form.

Solution: $y = \dfrac{3x-2}{(x-1)^{\frac{1}{2}}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}}\times 3 - (3x-2)\times\frac{1}{2}(x-1)^{\frac{-1}{2}}}{x-1}$$

write without negative indices

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}}\times 3 - \dfrac{(3x-2)}{2(x-1)^{1/2}}}{x-1}$$

multiply top and bottom by $2(x-1)^{1/2}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)^{\frac{1}{2}}\times 3 - \dfrac{(3x-2)}{2(x-1)^{1/2}}}{x-1} \times \frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6(x-1) - (3x-2)}{2(x-1)^{3/2}} = \frac{6x-6-3x+2}{2(x-1)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x-4}{2(x-1)^{3/2}}$$

Derivatives of e^x and $\log_e x \equiv \ln x$.

$y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x$

$y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$

$y = \ln kx \quad \Rightarrow \quad y = \ln k + \ln x \quad \Rightarrow \quad \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$

or $y = \ln kx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{kx} \times k = \frac{1}{x}$ using the chain rule

$y = \ln x^k \quad \Rightarrow \quad y = k \ln x \quad \Rightarrow \quad \frac{dy}{dx} = k \times \frac{1}{x} = \frac{k}{x}$

Example: Find the derivative of $f(x) = x^3 - 5e^x$ at the point where $x = 2$.

Solution: $f(x) = x^3 - 5e^x$

$\Rightarrow \quad f'(x) = 3x^2 - 5e^x$

$\Rightarrow \quad f'(2) = 12 - 5e^2 = -24{\cdot}9$

Example: Differentiate the function $f(x) = \ln 3x - \ln x^5$

Solution: $f(x) = \ln 3x - \ln x^5 = \ln 3 + \ln x - 5 \ln x = \ln 3 - 4 \ln x$

$\Rightarrow \quad f'(x) = \frac{-4}{x}$

or we can use the chain rule

$\Rightarrow \quad f'(x) = \frac{1}{3x} \times 3 - \frac{1}{x^5} \times 5x^4 = \frac{-4}{x}$

Example: Find the derivative of $f(x) = \log_{10} 3x$.

Solution: $f(x) = \log_{10} 3x = \frac{\ln 3x}{\ln 10}$ using change of base formula

$= \frac{\ln 3 + \ln x}{\ln 10} = \frac{\ln 3}{\ln 10} + \frac{\ln x}{\ln 10}$

$\Rightarrow \quad f'(x) = 0 + \frac{\frac{1}{x}}{\ln 10} = \frac{1}{x \ln 10}$.

Example: $y = e^{x^2}$. Find $\frac{dy}{dx}$.

Solution: $y = e^u$, where $u = x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = e^u \times \frac{du}{dx} = e^u \times 2x = 2x e^{x^2}$.

Example: $y = \ln 7x^3$. Find $\frac{dy}{dx}$.

Solution: $y = \ln u$, where $u = 7x^3$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} = \frac{1}{7x^3} \times 21x^2 = \frac{3}{x}$.

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Using the chain rule we can see that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$, $\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Example: $x = \sin^2 3y$. Find $\frac{dy}{dx}$.

Solution: First find $\frac{dy}{dx}$ as this is easier.

$\frac{dy}{dx} = 2 \sin 3y \cos 3y \times 3 = 6 \sin 3y \cos 3y$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{6 \sin 3y \cos 3y} = \frac{1}{3 \sin 6y} = \frac{1}{3} \operatorname{cosec} 6y$

Derivative of a^x

$y = a^x \quad \Rightarrow \quad \frac{dy}{dx} = a^x \ln a$

You should learn this result.

Example: If $y = 5^{x+2}$, find $\frac{dy}{dx}$.

Solution: $y = 5^x \times 5^2 = 25 \times 5^x$

$\Rightarrow \quad \frac{dy}{dx} = 25 \times 5^x \ln 5$

Proof of the result

$y = a^x$

$\Rightarrow \quad \ln y = \ln a^x = x \ln a$

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln a$

$\Rightarrow \quad \frac{dy}{dx} = y \ln a = a^x \ln a$

You should **know** this proof.

Trigonometric differentiation

x must be in RADIANS when differentiating trigonometric functions.

$f(x)$	$f'(x)$	*important formulae*
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	$\sin^2 x + \cos^2 x = 1$
$\tan x$	$\sec^2 x$	
$\sec x$	$\sec x \tan x$	$\tan^2 x + 1 = \sec^2 x$
$\cot x$	$-\text{cosec}^2 x$	
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$	$1 + \cot^2 x = \text{cosec}^2 x$

Chain rule – further examples

Example: $y = \sin^4 x$. Find $\frac{dy}{dx}$.

Solution: $y = \sin^4 x$. Put $u = \sin x$

$\Rightarrow \quad y = u^4$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = 4u^3 \times \frac{du}{dx} = 4\sin^3 x \times \cos x$.

Example: $y = e^{\sin x}$. Find $\frac{dy}{dx}$.

Solution: $y = e^u$, where $u = \sin x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = e^u \times \frac{du}{dx} = e^{\sin x} \times \cos x = \cos x \times e^{\sin x}$.

Example: $y = \ln(\sec x)$. Find $\frac{dy}{dx}$.

Solution: $y = \ln u$, where $u = \sec x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} = \frac{1}{\sec x} \times \sec x \tan x = \tan x .$$

Trigonometry and the product and quotient rules

Example: Differentiate $y = x^2 \times \operatorname{cosec} 3x$.

Solution: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$y = x^2 \times \operatorname{cosec} 3x$

Put $u = x^2$ and $v = \operatorname{cosec} 3x$

$$\frac{dy}{dx} = x^2 \times (-3\operatorname{cosec} 3x \cot 3x) + \operatorname{cosec} 3x \times 2x$$

$$= -3x^2 \operatorname{cosec} 3x \cot 3x + 2x \operatorname{cosec} 3x.$$

Example: Differentiate $y = \frac{\tan 2x}{7x^3}$

Solution: $y = \frac{u}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$y = \frac{\tan 2x}{7x^3}$

Put $u = \tan 2x$ and $v = 7x^3$

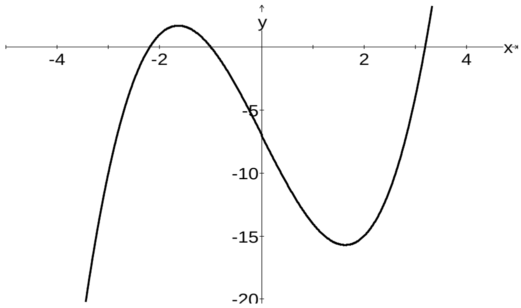
$$\Rightarrow \quad \frac{dy}{dx} = \frac{7x^3 \times 2\sec^2 2x - \tan 2x \times 21x^2}{(7x^3)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{14x^3 \sec^2 2x - 21x^2 \tan 2x}{49x^6}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x\sec^2 2x - 3\tan 2x}{7x^4}$$

6 Numerical methods

Locating the roots of f(x) = 0



A quick sketch of the graph of $y = f(x)$ can give a rough idea of the roots of $f(x) = 0$.

If $y = f(x)$ **changes sign** between $x = a$ and $x = b$ **and** $f(x)$ is **continuous** in this region then a root of $f(x) = 0$ lies between $x = a$ and $x = b$.

The iteration $x_{n+1} = g(x_n)$

Example:

(*a*) Show that a root, α, of the equation $f(x) = x^3 - 8x - 7 = 0$ lies between 3 and 4.

(*b*) Show that the equation $x^3 - 8x - 7 = 0$ can be re–arranged as $x = \sqrt[3]{8x+7}$.

(*c*) Starting with $x_1 = 3$, use the iteration $x_{n+1} = \sqrt[3]{8x_n + 7}$ to find the first four iterations for x.

(*d*) Show that your value of x_4 is correct to 3 S.F.

Solution:

(*a*) $f(3) = 27 - 24 - 7 = -4$, and $f(4) = 64 - 32 - 7 = +25$

Thus $f(x)$ **changes sign** and $f(x)$ is **continuous** $\Rightarrow$ there is a root between 3 and 4.

(*b*) $x^3 - 8x - 7 = 0 \quad \Rightarrow \quad x^3 = 8x + 7 \quad \Rightarrow \quad x = \sqrt[3]{8x+7}$.

(*c*) $x_1 = 3$

$\Rightarrow \quad x_2 = \sqrt[3]{8\times3+7} \quad \Rightarrow \quad = 3{\cdot}14138065239$

$\Rightarrow \quad x_3 = 3{\cdot}17912997899$

$\Rightarrow \quad x_4 = 3{\cdot}18905898325$

(*d*) $x_4 = 3{\cdot}19$ to 3 S.F.

$f(3{\cdot}185) = 3{\cdot}185^3 - 8 \times 3{\cdot}185 - 7 = -0{\cdot}17 \ldots$

$f(3{\cdot}195) = 3{\cdot}195^3 - 8 \times 3{\cdot}195 - 7 = +0{\cdot}05 \ldots$

$f(x)$ **changes sign** and $f(x)$ is **continuous**

$\Rightarrow$ there is a root in the interval [3·185, 3·195]

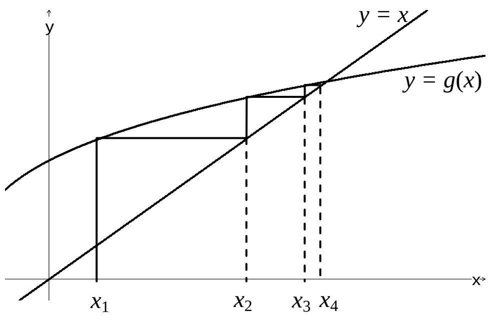
$\Rightarrow \quad \alpha = 3{\cdot}19$ to 3 S.F.

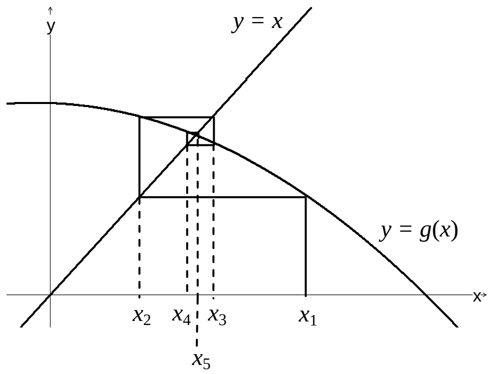
Conditions for convergence

If an equation is rearranged as $x = g(x)$ and if there is a root $x = \alpha$

then the iteration $x_{n+1} = g(x_n)$, starting with an approximation near $x = \alpha$

(i) will converge if $-1 < g'(\alpha) < 1$



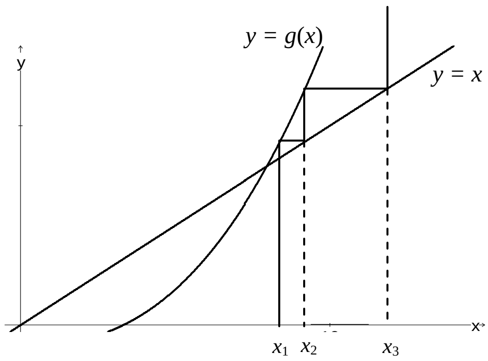


(a) will converge without oscillating if $0 < g'(\alpha) < 1$,

(b) will oscillate and converge if $-1 < g'(\alpha) < 0$,

(ii) will diverge if

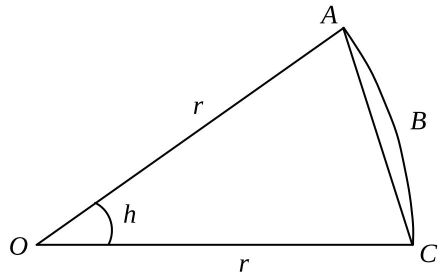
$g'(\alpha) < -1$ or $g'(\alpha) > 1$.



7 Appendix

Derivatives of sin x and cos x

$$\lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$$



If h is a small angle,

the area of the triangle *OAC*, $\frac{1}{2}r^2\sin h$,

will be approximately the same as

the area of the sector *OABC*, $\frac{1}{2}r^2h$

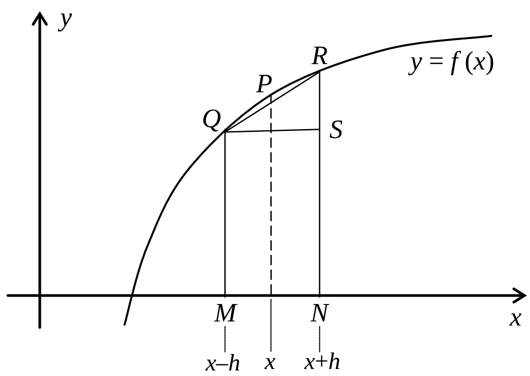
$$\Rightarrow \quad \frac{\frac{1}{2}r^2\sin h}{\frac{1}{2}r^2 h} \cong 1$$

$$\Rightarrow \quad \frac{\sin h}{h} \cong 1$$

as $h \to 0$, we have $\lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 1$.

Note that the formula for the area of sector is only true if h is in **radians.**

Alternative formula for derivative



The gradient of the curve at P will be nearly equal to the gradient of the line QR.

$QM = f(x - h)$ and $RN = f(x + h)$

$\Rightarrow \quad RS = f(x + h) - f(x - h)$

$QS = MN = 2h$

$\Rightarrow$ gradient of $QR = \dfrac{f(x+h) - f(x-h)}{2h}$

and as $h \to 0$, the gradient of $QR \to f'(x)$ = gradient of the curve at P

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h}$$

In C2 we used the formula $f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

Derivatives of sin *x* and cos *x*

$f(x) = \sin x$ and $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x-h)}{2h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - (\sin x \cos h - \cos x \sin h)}{2h}$$

$$= \lim_{h \to 0} \frac{2\cos x \sin h}{2h}$$

$$= \lim_{h \to 0} \cos x \frac{\sin h}{h}$$

but $\lim_{h \to 0} \left(\frac{\sin h}{h}\right) = 1$

$$\Rightarrow f'(x) = \frac{d}{dx}(\sin x) = \cos x$$

Similarly, we can show that $\frac{d}{dx}(\cos x) = -\sin x$

***x* must be in RADIANS.**

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