M2 Jan 2011 Written Solutions

① (a) $\underline{r} = \int \underline{v}\, dt = \int (4+3t^2)\underline{i} + \int (12-8t)\underline{j}\ dt$

$$\underline{r} = (4t + t^3)\underline{i} + (12t - 4t^2)\underline{j} + \underline{c}$$

Use $r = 5\underline{i} - 7\underline{j}$ when $t = 0$

so i: $5 = 0 + c_i \Rightarrow c_i = 5$

j: $-7 = 0 + c_j \Rightarrow c_j = -7$

$$\underline{r} = (4t + t^3 + 5)\underline{i} + (12t - 4t^2 - 7)\underline{j}$$

(b) $\underline{a} = \frac{dv}{dt} = (6t)\underline{i} + (-8)\underline{j}$

$\underline{a} = 6t\underline{i} - 8\underline{j}$

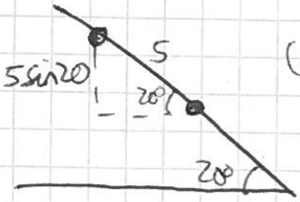
(c) $F = m\underline{a}$ $\qquad \underline{a}$ at $t = 1$:

$\underline{a} = 6\underline{i} - 8\underline{j}$

$F = 2(6\underline{i} - 8\underline{j}) = 12\underline{i} - 16\underline{j}$

magnitude: $\sqrt{12^2 + 16^2} = 20\ N$

②



(a) $\Delta GPE = mg\Delta h = 4 \times 9.8 \times 5\sin 20$

$= 67.0354$

$\underline{\underline{67.0\ J}}$

(b) Smooth $\Rightarrow$ no wd against friction so

$\Delta GPE = \Delta E_k$

$\Rightarrow \underline{\underline{67.0\ J}}$

(c) $67.0359 = \frac{1}{2}mv^2$

$$\Rightarrow \quad v = \sqrt{\frac{2 \times 67.0359}{4}}$$

$$v = 5.78947$$

$$v = 5.79\ ms^{-1}$$

③ (a) Water so 400 litres is 400 kg

(Since $1000\ L = 1\ m^3$ and density of water $= 1000\ kg/m^3$)

so $mg\Delta h$

$= 400 \times 9.8 \times 8$

$= 31360\ J$

(b) $\frac{1}{2}mv^2 = \frac{1}{2} \times 400 \times 2^2 = 800\ J$

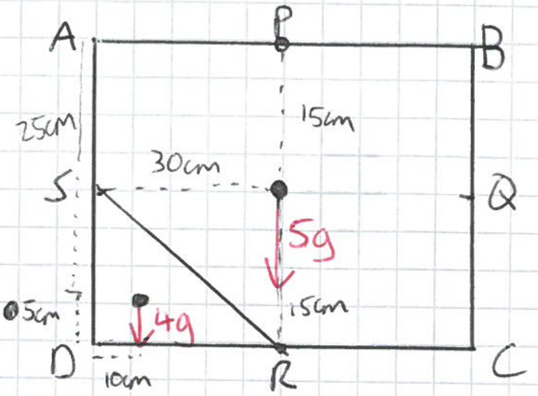
(c) Gained energy $= 31360 + 800\ J$

$= 32160\ J$

This is in one minute $= 60s$

$P = \frac{E}{t} = \frac{32160}{60} = 536\ W$

④



(a) total mass $5+4=9kg$

x coordinate only → taking A as the origin

$$9g\bar{x} = 10\times 4g + 30\times 5g$$

$$9g\bar{x} = 40g + 150g = 190g$$

$$9\bar{x} = 190$$

$$\bar{x} = 21.\dot{1}$$

$$\bar{x} = \underline{21.1\ cm}$$

(b) Use A as the origin again

$$9g\bar{y} = 15\times 5g + 25\times 4g$$

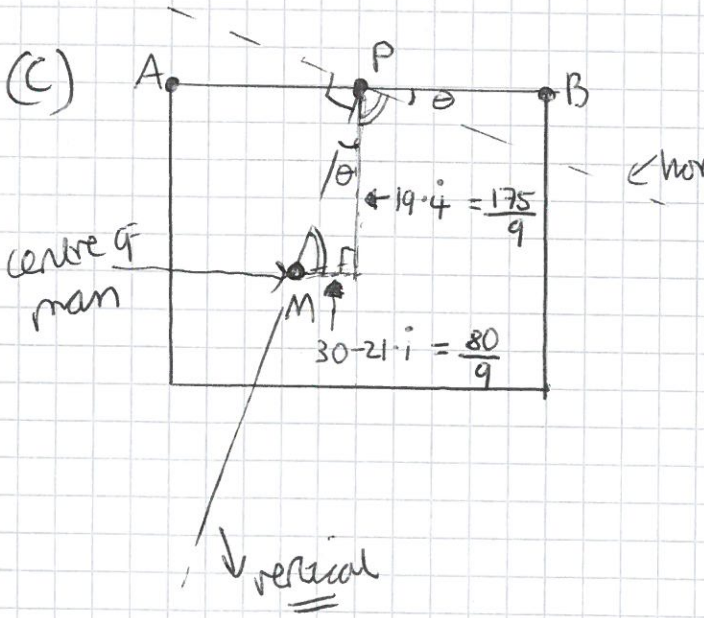
$$9g\bar{y} = 75g + 100g$$

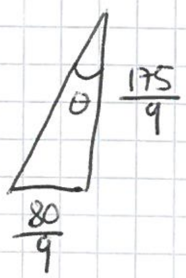
$$9\bar{y} = 175$$

$$\bar{y} = 19.\dot{4}$$

$$\bar{y} = \underline{19.4\ cm}$$

(C)





$$\tan\theta = \frac{\frac{80}{9}}{\frac{175}{9}} = \frac{80}{175}$$

$$\theta = \tan^{-1}\left(\frac{80}{175}\right)$$

$$= 24.567$$

$$\theta = 24.6^\circ$$

(d) Need to have $\bar{x}$ directly at 30cm from A (so COM is directly below P).

Total mass $= 9+m$

$\Rightarrow (9+m)\cancel{g}\bar{x} = 10\times4\cancel{g} + 30\times5\cancel{g} + 60\times m\cancel{g}$

$\Rightarrow (9+m)\times30 = 40 + 150 + 60m$

$\Rightarrow 80 = 30m \qquad \Rightarrow m = \frac{8}{3}$ kg

(e) Centre of mass in middle of rectangular lamina.

(5) (a) $F = mr\omega^2$

resolving vertically

$\left(\begin{array}{ll} F = \mu R & R = mg \\ \Rightarrow F = \mu mg = & 0.3mg \end{array}\right)$

so $\quad 0.3\cancel{m}g = \cancel{m}r\omega^2 \qquad r = 0.8m$

$\sqrt{\frac{0.3g}{0.8}} = \omega$

$\Rightarrow \omega = 1.917 \qquad \omega = 1.92\ \text{rads}^{-1}$

(b) (i) 45 rev min^{-1}

$45 \times 2\pi = 90\pi$ rad min^{-1}

$\div 60 \qquad \frac{3}{2}\pi$ rads^{-1}

(ii) $F = mr\omega^2$

$= m\times0.15\times\left(\frac{3}{2}\pi\right)^2$

$F = \mu R$ and resolving vertically gives $R = mg$

so $F = \mu mg$

$\Rightarrow \mu\cancel{m}g = \cancel{m}\times0.15\times\frac{9}{4}\pi^2$

$\mu = \frac{0.15\times\frac{9}{4}\pi^2}{g}$

$\mu = 0.340$

⑥ (a)

$G_K \rightarrow GPE$

$\Delta GPE = mg\Delta h = mg \times 2a$

$= 2amg$

$$\Rightarrow \quad \frac{1}{2}\cancel{m}(5v)^2 = \frac{1}{2}\cancel{m}(3v)^2 + 2a\cancel{m}g$$

$$\frac{25}{2}v^2 = \frac{9}{2}v^2 + 2ag$$

$$\frac{16}{2}v^2 = 2ag \quad \Rightarrow \quad v^2 = \frac{ag}{4}$$

$$v = \frac{\sqrt{ag}}{2}$$

(b) At B: $T_1 - mg = \frac{mv^2}{r}$ (Max tension)

$$T_1 - 4g = \frac{4(5v)^2}{a}$$

At A: $T_2 + mg = \frac{mv^2}{r}$ (min tension)

$$T_2 + 4g = \frac{4(3v)^2}{a}$$

$$T_1 = 4g + \frac{100v^2}{a} = 4g + \frac{100}{\cancel{a}} \times \frac{\cancel{a}g}{4}$$

$$= g(4 + 25) = 29g$$

$$T_2 = -4g + \frac{4 \times 9v^2}{a} = -4g + \frac{36}{\cancel{a}} \times \frac{\cancel{a}g}{4}$$

$$= -4g + 9g = 5g$$

$29g : 5g$

$= 29 : 5$

7 (a) $WD = \int F \, dx$ $\qquad F = \frac{7x}{l}$

extension (x) goes from 0 to e

so $WD = \int_0^e \frac{7x}{l} \, dx$

$$= \left[\frac{7x^2}{2l}\right]_0^e = \frac{7e^2}{2l} - 0$$

$$= \frac{7e^2}{2l}$$

(b)(i) In equilibrium so $T = mg = 7g$

$7g = \frac{7x}{l} \implies 7g = \frac{196x}{2}$

$14g = 196x$

$\implies \frac{g}{14} = x = \underline{\underline{0.7m}}$

(ii) At rest $\implies$ ~~all~~ some EPE stored is converted to GPE

EPE stored $= \frac{7x^2}{2l} = \frac{196 \times 0.7^2}{2 \times 2} = 24.01J$

Say that it has moved up by x metres

then $e = (0.7 - x)$

so $EPE = \frac{196 \times (0.7-x)^2}{2 \times 2}$ and $\Delta GPE = mgx$

$= 49[x^2 - 1.4x + 0.49]$

$= 49x^2 - 68.6x + 24.01$

so $\cancel{24.01} = 49x^2 - 68.6x + \cancel{24.01} + mgx$

$0 = 49x^2 + (mg - 68.6)x$

$\underline{\underline{m=4}}$

$\Rightarrow \quad 0 = 49x^2 - 29.4x \qquad \Rightarrow \quad 0 = x(49x - 29.4)$

so $\underline{\underline{x = 0}}$

or $49x - 29.4 = 0$

$\Rightarrow x = \dfrac{29.4}{49} = 0.6$

$x = 0.6m$

$\Rightarrow$ extension $= 0.7 - 0.6 = \underline{\underline{0.1m}}$

(iii) speed is max $\therefore$ constant

so $a = 0$ and $F = 0$

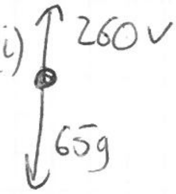
so $T = 4g$

$4g = \dfrac{7x}{l} = \dfrac{196x}{2}$

$8g = 196x \qquad \Rightarrow \quad x = \dfrac{8 \times 9.8}{196}$

$= \underline{\underline{0.4m}}$

(8) (a)(i)



$$F = 65g - 260v \quad = 65\left(9 - \frac{260}{65}v\right)$$
$$= 65(9.8 - 4v) \text{ N}$$

(ii) $a = \frac{dv}{dt}$ $\quad a = \frac{F}{m} \quad \Rightarrow \quad a = 9.8 - 4v$
$$= -4\left(v - \frac{9.8}{4}\right) = -4(v - 2.45)$$

So $\frac{dv}{dt} = -4(v - 2.45)$

(b) Separate variables.

$$\int \frac{1}{v - 2.45} = \int -4 \, dt$$

$$\Rightarrow \quad \ln|v - 2.45| = -4t + C$$
$$v - 2.45 = e^{-4t + C}$$
$$v = e^{-4t}e^{c} + 2.45$$
$$= Ae^{-4t} + 2.45$$

Use $t = 0$ $\quad v = 19.6$

$$\Rightarrow \quad 19.6 = Ae^{0} + 2.45$$
$$19.6 - 2.45 = A$$
$$17.15 = A$$

$$\Rightarrow \quad v = 17.15e^{-4t} + 2.45$$