FP1 - June 2014

(1)

$$y_{n+1} = y_n + h\,f(x)$$

$x_1 = 9$ $\quad \rightarrow \quad 6 + 0.25 \times \left[\frac{1}{2} + \sqrt{9} \right]$

$y_1 = 6$ $\quad = \quad 6.05$

$h = 0.25$

$x_2 = 9.25$ $\quad \rightarrow \quad 6.05 + 0.25 \times \left[\frac{1}{2} + \sqrt{9.25} \right]$

$y_2 = 6.05$ $\quad = \quad 6.0995895\ldots$

$= 6.0996$ (4dp)

(2) a) $\alpha + \beta = -8/2 = -4 \qquad \alpha\beta = 1/2$

b) i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (-4)^2 - 2(1/2)$

$= 16 - 1 = 15$

ii) $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$

$= 15^2 - 2(\alpha\beta)^2$

$= 225 - 2(1/4) = 449/2$

c) $\boxed{\text{Sum}}$ $\quad 2\alpha^4 + 1/\beta^2 + 2\beta^4 + 1/\alpha^2$

$= 2\alpha^4 + 2\beta^2 + 1/\alpha^2 + 1/\beta^2$

$= 2(\alpha^4 + \beta^4) + \beta^2/\alpha^2\beta^2 + \alpha^2/\alpha^2\beta^2$

$= 2(\alpha^4 + \beta^4) + \dfrac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

$= 2(449/2) + 15/(1/2)^2$

$= 449 + 60 = 509.$

$\boxed{\text{PRODUCT}}$ $\left(2\alpha^4 + 1/\beta^2\right)\left(2\beta^4 + 1/\alpha^2\right)$

$= 4\alpha^4\beta^4 + 2\alpha^2 + 2\beta^4 + 1/\alpha^2\beta^2$

$= 4(\alpha\beta)^4 + 2(\alpha^2 + \beta^2) + 1/(\alpha\beta)^2$

$= 4(1/2)^4 + 2(15) + 1/(1/2)^2$

$= 1/4 + 30 + 4 = 137/4$

$\rightarrow x^2 - \boxed{\text{SUM}}\, x + \boxed{\text{PRODUCT}} = 0$

$\rightarrow x^2 - 509x + 137/4 = 0$

$\rightarrow 4x^2 - 2036x + 137 = 0$

(3) $\sum r^2(r-6) = \sum r^3 - 6r^2 = \sum r^3 - 6\sum r^2$

Need: $\sum_1^{60} r^3 - 6\sum_1^{60} r^2 - \left(\sum_1^{2} r^3 - 6\sum_1^{2} r^2\right)$

$= 1/4\,(60)^2(61)^2 - 6\; 1/6\,(60)(61)(2\times 60+1) - ((1+8) - (6+24))$

$= 3,348,900 - 442,860 - (-21)$

$= 2,906,061$

(4) Let $z = a + ib$

$\rightarrow 5i(a+ib) + 3(a-ib) + 1b = 8i$

$5ai + 5i^2ab + 3a - 3bi + 1b = 8i$

$5ai - 5ab + 3a - 3bi + 1b = 8i$

$\boxed{\text{REAL}}$ $-5b + 3a + 1b = 0$ ①

$\boxed{\text{IMAG}}$ $5a - 3b = 8$ ②

(1) $3a - 5b = -16 \quad \times 5 \rightarrow \quad 15a - 25b = -80$

(2) $5a - 3b = 8 \quad \times 3 \rightarrow \quad 15a - 9b = 24$

(2) - (1) $\rightarrow \quad 16b = 104 \quad \rightarrow \quad b = 6.5$

(1) $3a - 5(6.5) = -16$

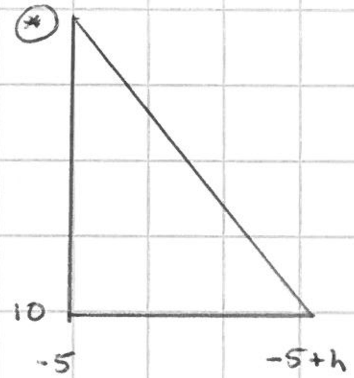
$3a - 32.5 = -16$

$3a = 16.5$

$a = 5.5$

$\rightarrow \quad z = 5.5 + 6.5i$

(5) a)



(*) $y = (-5+h)(-5+h+3)$

$= (-5+h)(-2+h)$

$= 10 - 5h - 2h + h^2$

$= h^2 - 7h + 10$

$$\text{Gradient} = \frac{h^2 - 7h + 10 - 10}{-5 + h - (-5)}$$

$$= \frac{h^2 - 7h}{h} = h - 7$$

b) As $h \rightarrow 0$, gradient $\rightarrow 0 - 7 = -7$

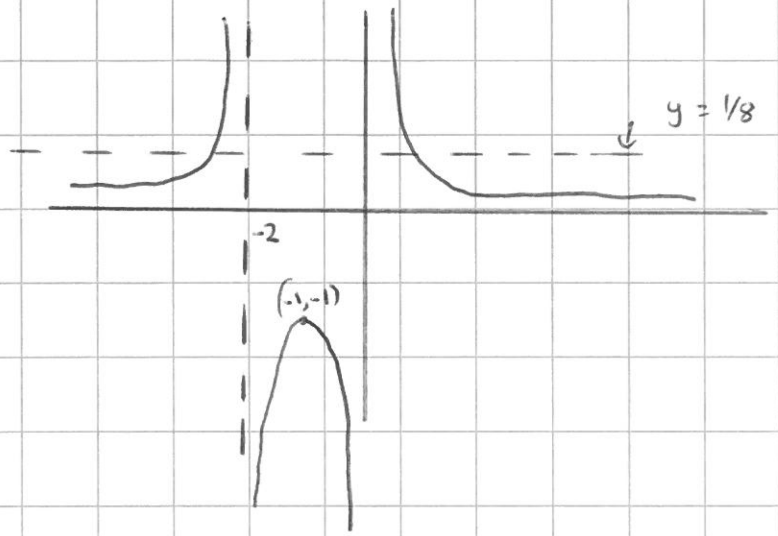
This is a good approximation of gradient of curve at $(-5, 10)$

6) a) $y = \dfrac{1}{x(x+2)}$ $\qquad \boxed{x = 0}, \ \boxed{x = -2}$

As $x \to \infty$, $y \to 1/\infty \Rightarrow \boxed{y = 0}$

b) i) $x = -1 \to y = 1/_{-1(-1+2)} = 1/_{-1} = -1$

ii)



c) Mark $y = 1/8$ on sketch

Solve:

$1/_{x(x+2)} = 1/8$

$8 = x(x+2)$

$8 = x^2 + 2x$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

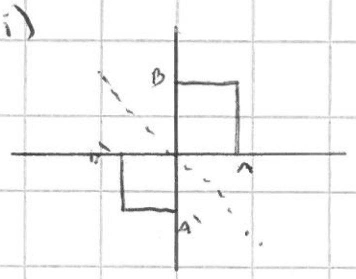
$x = -4 \qquad x = 2$

From sketch, we need curve <u>below</u> $y = 1/8$

$\to x \leq -4, \quad x \geq 2$

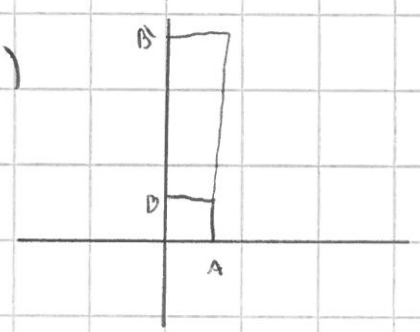
or $-2 < x < 0$

7) a) i)



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

ii)



$$\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

b) Change order: →

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ -7 & 0 \end{bmatrix}$$

c) i)

$$\begin{bmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} \quad \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \quad = \quad 12\,I$$

ii) Scale Factor must be $\sqrt{12} = 2\sqrt{3}$

Take $2\sqrt{3}$ out of matrix:

$$2\sqrt{3} \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

Reflection is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$\sin^{-1}(-1/2) = -30$ or -150 → $\theta = -15°$ or $\theta = -75°$

Check in cos: $\cos(2\theta)$ must $= -\sqrt{3}/2$

$\cos(-30) = \sqrt{3}/2$ ×

$\cos(-150) = -\sqrt{3}/2$ ✓

$\therefore \theta = -75$

$\rightarrow$ Line of reflection: $y = [\tan(-75)]x$

or $y = [-\tan(75)]x$

or $y = [\tan(105)]x$

(8) a) $\theta = 2n\pi \pm a$

Key angle: $\cos^{-1}(\sqrt{3}/2) = \pi/4$

$$\rightarrow \tfrac{5}{4}x - \pi/3 = 2n\pi \pm \pi/4$$

$$\tfrac{5}{4}x = 2n\pi + \pi/3 \pm \pi/4$$

$$5x = 8n\pi + 4\pi/3 \pm \pi$$

$$x = \tfrac{8}{5}n\pi + \tfrac{4\pi}{15} \pm \pi/5$$

$$\cancel{x = \tfrac{8}{5}n\pi + 20\pi/3 + 5\pi}, \quad \cancel{x = \tfrac{8}{5}n\pi + 20\pi/3 - 5\pi}$$

$$x = \frac{24n\pi}{15} + \frac{4\pi}{15} + \frac{3\pi}{15}, \quad x = \frac{24n\pi}{15} + \frac{4\pi}{15} - \frac{3\pi}{15}$$

$$x = \frac{24n\pi + 7\pi}{15} \qquad x = \frac{24n\pi + \pi}{15}$$

b) Smallest value of $n = 0 \rightarrow \frac{7\pi}{15}$ or $\frac{\pi}{15}$

Largest value of $n = 12 \rightarrow 10\frac{2}{3}\pi$ or $19.2666\ldots\pi$

Need $\sum_0^{12} \frac{24n\pi + 7\pi}{15} + \sum_0^{12} \frac{24n\pi + \pi}{15}$

$= \sum_0^{12} \frac{24n\pi}{15} + \sum_0^{12} \frac{7\pi}{15} + \sum_0^{12} \frac{24n\pi}{15} + \sum_0^{12} \frac{\pi}{15}$

$=$ use $\sum_0^n r = \frac{1}{2}n(n+1)$ & $\sum_0^n 1 = n+1$ ← As it started 0.

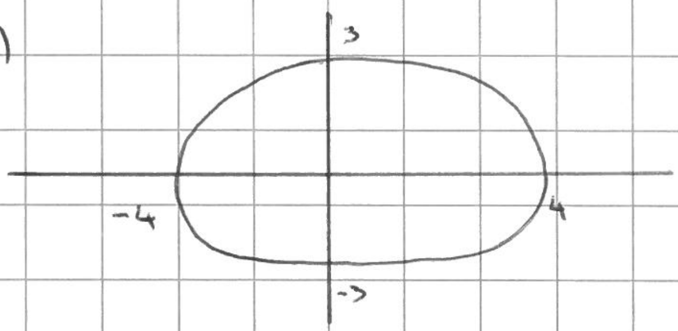
$\rightarrow \frac{24\pi}{15}\sum_0^{12} n + \frac{7\pi}{15}\sum_0^{12} 1 + \frac{24\pi}{15}\sum_0^{12} n + \frac{\pi}{15}\sum_0^{12} 1$

$= \left[\frac{24\pi}{15} \times \frac{1}{2} \times 12 \times 13\right] + \frac{7\pi}{15}(13) + \left[\frac{24\pi}{15} \times \frac{1}{2} \times 12 \times 13\right] + \frac{\pi}{15} \times 13$

$= \frac{\pi}{15}\left[1872 + 91 + 1872 + 13\right]$

$= 3848/15\ \pi \quad \rightarrow k = 3848/15$

(9) a)



b) $y = x + k$

$\rightarrow \frac{x^2}{16} + \frac{(x+k)^2}{9} = 1$

$\rightarrow 9x^2 + 16(x+k)^2 = 144$

$9x^2 + 16\left[x^2 + 2xk + k^2\right] = 144$

$9x^2 + 16x^2 + 32xk + 16k^2 - 144 = 0$

$25x^2 + 32kx + (16k^2 - 144) = 0$

For 2 distinct solutions, $b^2 - 4ac > 0$

$$\rightarrow (32k)^2 - 4 \times 25 \times (16k^2 - 144) > 0$$

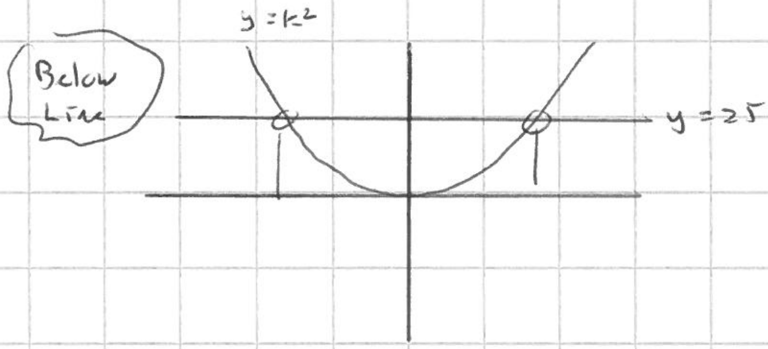
$$1024k^2 - 1600k^2 + 14400 > 0$$

$$\cancel{14400 > 1600k^2}$$

$$-576k^2 + 14400 > 0$$

$$14400 > 576k^2$$

$$k^2 < 25$$



$\therefore \quad -5 < k < 5$

c) Translation $\rightarrow \dfrac{(x-a)^2}{16} + \dfrac{(y-b)^2}{9} = 1$

$$\rightarrow 9(x-a)^2 + 16(y-b)^2 = 144$$

$$\rightarrow 9(x^2 - 2ax + a^2) + 16(y^2 - 2by + b^2) = 144$$

$$\rightarrow 9x^2 - 18ax + 9a^2 + 16y^2 - 32by + 16b^2 = 144$$

$$9x^2 + 16y^2 \boxed{-18ax} \boxed{-32by} = \boxed{144 - 9a^2 - 16b^2}$$

NEED: $9x^2 + 16y^2 \boxed{+18x} \boxed{-64y} \boxed{= c}$

① ② ③

① $a = -1$

② $b = 2$

③ $c = 144 - 9(-1)^2 - 16(2)^2 = 71$

d) From part b, we know that $b^2 - 4ac = 0$

when $k = 5$ or $k = -5$

$\rightarrow$ $y = x + 5$ or $y = x - 5$ are tangents

to original curve E

E has been translated $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\therefore$ New tangents are:

$(y - 2) = (x + 1) + 5$

$y - 2 = x + 6$

$y = x + 8$

$(y - 2) = (x + 1) - 5$

$y - 2 = x - 4$

$y = x - 2$