Further Pure 1 - January 2013

① Euler: $y_{n+1} = y_n + hf(x_n)$

$y_1 = 3$

$x_1 = 1$ $\quad y_2 = 3 + 0.1 \times \left[\frac{1}{1+1^3}\right]$

$h = 0.1$

$= 3.05$

$y_2 = 3.05$

$x_2 = 1.1$ $\quad y_3 = 3.05 + 0.1 \times \left[\frac{1.1}{1+1.1^3}\right]$

$h = 0.1$

$= 3.097190...$

$= 3.0972$ (4dp)

② a) $w^2 + 6w + 34 = 0$

$$w = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 34}}{2}$$

$$= \frac{-6 \pm \sqrt{-100}}{2} = \frac{-6 \pm 10i}{2} = -3 \pm 5i$$

b) i) $z = i(1+i)(2+i)$

$= (i-1)(2+i) = 2i - 1 - 2 - i = -3 + i$

ii) $z + mz^* = ni$

$(-3+i) + m(-3-i) = ni$

$-3 - 3m + i - mi = ni$

$\boxed{\text{REAL}}$ $-3 - 3m = 0$

$\rightarrow m = -1$

$\boxed{\text{IMG}}$ $i - mi = ni$

$\rightarrow i + i = ni \rightarrow n = 2$

③ a) SIN: $\theta = 2n\pi + \alpha$, $\quad \theta = 2n\pi + (\pi - \alpha)$

Key angle $(\alpha) = \sin^{-1}(\sqrt{3}/2) = \pi/3$

$\rightarrow 2x + \pi/4 = 2n\pi + \pi/3$, $\quad 2x + \pi/4 = 2n\pi + 2\pi/3$

$2x = 2n\pi + \pi/12$, $\quad 2x = 2n\pi + 5\pi/12$

$\therefore x = n\pi + \pi/24 \quad , \quad x = n\pi + 5\pi/24$

b) Try $n = 5 \rightarrow x = 5\pi + \pi/24 = 5\tfrac{1}{24}\pi$

or $x = 5\pi + 5\pi/24 = 5\tfrac{5}{24}\pi$

$\therefore n = 5$ is answer

(4) $$\int_{25}^{\infty} \frac{1}{x\sqrt{x}}\,dx = \int_{25}^{p} x^{-3/2}\,dx$$

$$= \left[-2x^{-1/2}\right]_{25}^{p} = \left[\frac{-2}{\sqrt{x}}\right]_{25}^{p}$$

$$= \left(\frac{-2}{\sqrt{p}}\right) - \left(\frac{-2}{\sqrt{25}}\right)$$

$$= 2/5 - \frac{2}{\sqrt{p}}$$

As $p \rightarrow \infty$, $-2/\sqrt{p} \rightarrow 0$

$\therefore \int \rightarrow 2/5$

(5) a) $\alpha + \beta = -2 \quad , \quad \alpha\beta = -5$

b) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2\times(-5) = 14$

c) $\boxed{\text{SUM}}$ $\alpha^3\beta + 1 + \alpha\beta^3 + 1$

$= 2 + \alpha^3\beta + \alpha\beta^3$

$= 2 + \alpha\beta(\alpha^2 + \beta^2)$

$= 2 + (-5)(14) = -68$

$\boxed{\text{PRODUCT}}$ $(\alpha^3\beta + 1)(\alpha\beta^3 + 1)$

$= \alpha^4\beta^4 + \alpha^3\beta + \alpha\beta^3 + 1$

$= (\alpha\beta)^4 - 70 + 1$

$= (-5)^4 - 70 + 1 = 556$

$\Rightarrow \quad x^2 - \boxed{\text{SUM}}\,x + \boxed{\text{PRODUCT}} = 0$

$\Rightarrow \quad x^2 + 68x - 556 = 0$

(6) a) i) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow m = 7$

ii) $X^3 = X^2 \times X$

$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

$\begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 13 & 14 \\ 21 & 6 \end{bmatrix}$ $\quad 7X = \begin{bmatrix} 7 & 14 \\ 21 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 13 & 14 \\ 21 & 6 \end{bmatrix} - \begin{bmatrix} 7 & 14 \\ 21 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 6I$

b) i) Reflection in x-axis

ii) $\Rightarrow \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$= 1/\sqrt{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

iii) $[A][B][\text{POINT}]$

$AB = 1/\sqrt{2} \times \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

Then $1/\sqrt{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$= 1/\sqrt{2} \times \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = 1/\sqrt{2} \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$= (-3/\sqrt{2}, \; -1/\sqrt{2})$$

(7) a) $y = ax^n$

$\log_{10} y = \log_{10} (ax^n)$

$\log_{10} y = \log_{10} (a) + n \log_{10} (x)$

$\rightarrow Y = \log_{10} (a) + n X$ = Linear relationship

b) y-Intercept = 4 $\rightarrow \log_{10} (a) = 4$

$\rightarrow a = 10^4 = 10,000$

Gradient $= -4/6 = -2/3 \rightarrow n = -2/3$

(8) a) $\rightarrow \sum 4r^3 - 6r^2 - 2r$

$= 4 \sum r^3 - 6 \sum r^2 - 2 \sum r$

$= n^2(n+1)^2 - n(n+1)(2n+1) - n(n+1)$

$= n(n+1) \left[n(n+1) - (2n+1) - 1 \right]$

$= n(n+1) \left[n^2 + n - 2n - 1 - 1 \right]$

$= n(n+1)(n^2 - n - 2)$

$= n(n+1)(n+1)(n-2)$

$= n(n-2)(n+1)^2 \rightarrow p = -2, \; q = 1$

b) $\sum_{11}^{20} = \sum_{1}^{20} - \sum_{1}^{10}$

$= 20(18)(21)^2 - 10(8)(11)^2 = 149,080$

(9) a) $y = 0 \rightarrow \frac{(x-4)^2}{4} = 1$

$\rightarrow (x-4)^2 = 4 \rightarrow x - 4 = \pm 2$

$\rightarrow x = 4 + 2$ or $x = 4 - 2$

$\rightarrow x = 6$ or $x = 2$

b) i) $y = mx \rightarrow \frac{(x-4)^2}{4} + (mx)^2 = 1$

$\rightarrow (x-4)^2 + 4m^2x^2 = 4$

$\rightarrow x^2 - 8x + 16 + 4m^2x^2 - 4 = 0$

$\rightarrow x^2 + 4m^2x^2 - 8x + 12 = 0$

$\rightarrow (1 + 4m^2)x^2 - 8x + 12 = 0$

ii) At tangent $b^2 - 4ac = 0$

$\rightarrow (-8)^2 - 4(1 + 4m^2)(12) = 0$

$\rightarrow 64 - 48 - 192m^2 = 0$

$\rightarrow 192m^2 = 16$

$\rightarrow m^2 = 1/12 \rightarrow m = 1/\sqrt{12}$ as $m > 0$

iii) $\boxed{\text{1st Find } x}$ $(1 + 4m^2)x^2 - 8x + 12 = 0$

$\rightarrow (1 + 4/12)x^2 - 8x + 12 = 0$

$4/3\, x^2 - 8x + 12 = 0$

$4x^2 - 24x + 36 = 0$

$x^2 - 6x + 9 = 0$

$(x - 3)(x - 3) = 0$

$\rightarrow x = 3$

$\boxed{\text{2nd. Find } y}$ $y = mx$

$\rightarrow y = 1/\sqrt{12} \times 3 = 3/\sqrt{12}$

$\therefore$ co-ordinates of $P = (3, 3/\sqrt{12})$