Further Pure 1 - June 2012

(1) a) $\alpha + \beta = 7/5 \qquad \alpha\beta = 1/5$

b) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2}{\alpha\beta} + \dfrac{\beta^2}{\alpha\beta}$

$= \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (7/5)^2 - 2(1/5) = 39/25$

$\therefore \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{39/25}{1/5} = 39/5$

c) $\boxed{\text{SUM}}$ $\alpha + 1/\alpha + \beta + 1/\beta = \alpha + \beta + 1/\alpha + 1/\beta$

$= (\alpha + \beta) + \dfrac{\alpha}{\alpha\beta} + \dfrac{\beta}{\alpha\beta}$

$= (\alpha + \beta) + \dfrac{\alpha + \beta}{\alpha\beta}$

$= 7/5 + \dfrac{7/5}{1/5} = 42/5$

$\boxed{\text{PRODUCT}}$ $(\alpha + 1/\alpha)(\beta + 1/\beta) = \alpha\beta + \alpha/\beta + \beta/\alpha + 1/\alpha\beta$

$= 1/5 + 39/5 + 1/\tfrac{1}{5}$

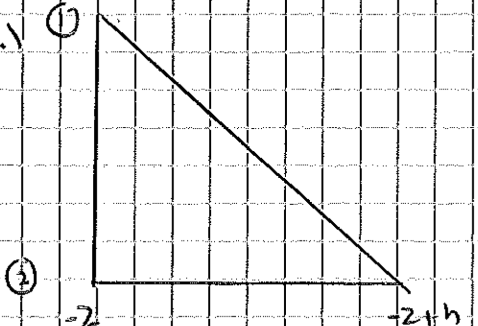
$= 13$

$x^2 - \boxed{\text{SUM}}\, x + \boxed{\text{PRODUCT}} = 0$

$\rightarrow x^2 - 42/5\, x + 13 = 0$

$\rightarrow 5x^2 - 42x + 65 = 0$

(2) a)



(1) $= (-2)^4 + (-2) = 14$

(2) $= (-2+h)^4 + (-2+h)$

$= (-2)^4 + 4(-2)^3 h + 6(-2)^2 h^2$

$+ 4(-2)h^3 + h^3 - 2 + h$

$= 16 - 32h + 24h^2 - 8h^3 + h^4 - 2 + h$

$= h^4 - 8h^3 + 24h^2 - 31h + 14$

Gradient $= \dfrac{h^4 - 8h^3 + 24h^2 - 31h + 14 - 14}{h}$

$= h^3 - 8h^2 + 24h - 31$

b) As $h \to 0$, grad $\to -31$

(3) a) $z = x + iy$

$\Rightarrow i(x + iy + 7) + 3(x - iy - i)$

$= xi - y + 7i + 3x - 3iy - 3i$

$= 3x - y + xi - 3iy + 4i$

$\boxed{\text{REAL}}$ $3x - y$

$\boxed{\text{IMAG}}$ $x - 3y + 4$

b) REAL + IMAG $= 0$

$\Rightarrow$ (1) $3x - y = 0$

(2) $x - 3y + 4 = 0$ or $x - 3y = -4$

(2) $\times 3 \Rightarrow$ $3x - 9y = -12$

(1) $-$ $3x - y = 0$

$-8y = -12 \Rightarrow y = 1.5$

(1) $3x - 1.5 = 0 \Rightarrow x = 0.5$

$\Rightarrow z = \frac{1}{2} + \frac{3}{2}i$

(4) SIN: $\theta = 360n + a$, $\theta = 360n + (180 - a)$

Key angle (a) $= \sin^{-1}(\cos(20)) = 70$

$70 - \frac{2}{3}x = 360n + 70$, $70 + \frac{2}{3}x = 360n + 110$

$-\frac{2}{3}x = 360n$, $+\frac{2}{3}x = 360n + 40$

$x = -\frac{3}{2}(360n)$, $x = -\frac{3}{2}(360n + 40)$

$x = +540n$ $x = -540n \mp 60$

(5) a) Put denominator → $x = -1$, $x = 2$

As $x \to \infty$, $y \to 0/1$ → $y = 0$

b) $y = -\frac{1}{2}$ → $-\frac{1}{2} = \dfrac{x}{(x+1)(x-2)}$

$$\to \quad -1 = \frac{2x}{(x+1)(x-2)}$$

$$-(x+1)(x-2) = 2x$$

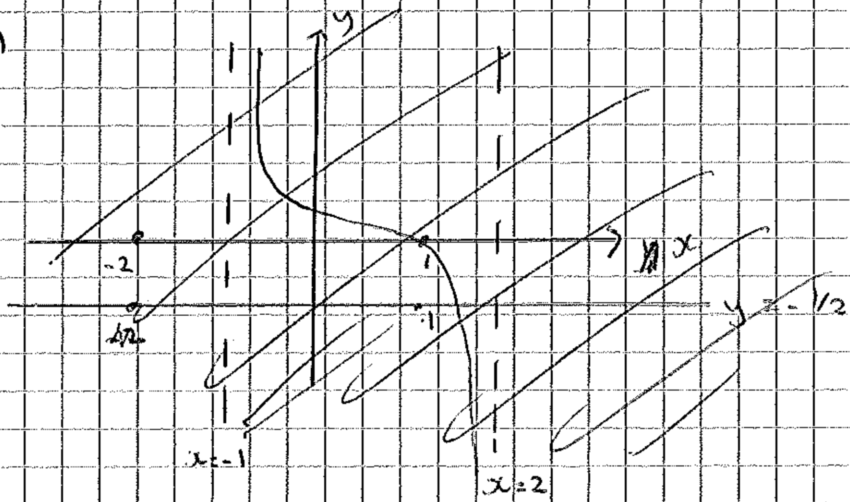
$$-x^2 + x + 2 = 2x$$

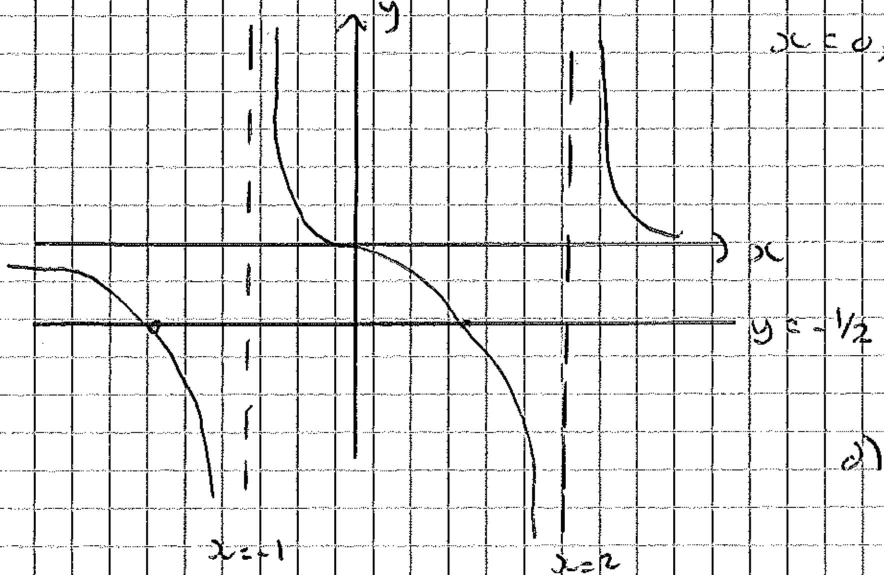
$$0 = x^2 + x - 2$$

$$(x+2)(x-1) = 0$$

$x = -2$ $\qquad$ $x = 1$

c)





$x = 0$, $y = 0$

d) curve below $y = -\frac{1}{2}$ when:

$-2 \leq x < -1$

$1 \leq x < 2$

(6) a) $\begin{bmatrix} \cos(135) & -\sin(135) \\ \sin(135) & \cos(135) \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

b) i) Looks like a) but with a factor of $\sqrt{2}$

$$\Rightarrow \sqrt{2}\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$\therefore$ Enlargement SF $\sqrt{2}$ & rotation 135^0 anti-clockwise

ii) M^2, scale factor $= (\sqrt{2})^2 = 2$

rotation $= 135 + 135 = 270^0$ anti-clockwise

iii) $M^2 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$M^4 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = -4I$$

iv) $M^{2012} = (M^4)^{503}$

$= (-4I)^{503}$

$= [-(2^2)I]^{503}$

$= -2^{1006} I^{503}$ ← $= I$ as I does not change

$= -2^{1006} I \Rightarrow n = 1006$

(7) a) Let $f(x) = 24x^3 + 36x^2 + 18x - 5$

$f(0.1) = 24(0.1)^3 + 36(0.1)^2 + 18(0.1) - 5 = -2.816$

$f(0.2) = 24(0.2)^3 + 36(0.2)^2 + 18(0.2) - 5 = 0.232$

sign change, $\therefore$ root between 0.1 & 0.2

b)

	Width	Mid-Point	Answer	
0.1 to 0.2	0.1	0.15	-1.409	(-ve)
0.15 to 0.2	0.05	0.175	-0.618875	(-ve)

$\therefore$ root lies between 0.175 & 0.2

c) Newton: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 0.2$

$f(x) = 24x^3 + 36x^2 + 18x - 5$

$f(0.2) = 0.232$

$f'(x) = 72x^2 + 72x + 18$

$f'(0.2) = 72(0.2)^2 + 72(0.2) + 18$
$= 32.25$

$\Rightarrow x_{n+1} = 0.2 - \frac{0.232}{32.25}$

$= 0.1934\ldots$ (4dp)

(8) a) $\frac{x^2}{5} + \frac{y^2}{4} = 1$

$\Rightarrow (\sqrt{5}, 0) \quad (-\sqrt{5}, 0) \quad (0, 2) \quad (0, -2)$

b) $\frac{(x-p)^2}{5} + \frac{y^2}{4} = 1$

c) $y = x + 4 \quad \Rightarrow \frac{(x-p)^2}{5} + \frac{(x+4)^2}{4} = 1$

$\times 20$

$4(x-p)^2 + 5(x+4)^2 = 20$

$4\left[x^2 - 2xp + p^2\right] + 5(x^2 + 8x + 16) = 20$

$\Rightarrow \quad 4x^2 - 8xp + 4p^2 + 5x^2 + 40x + 80 = 20$

$\Rightarrow \quad 9x^2 - 8xp + 40x + 4p^2 + 60 = 0$

$\Rightarrow \quad 9x^2 - (8p - 40)x + (4p^2 + 60) = 0$

b) For tangent: $b^2 - 4ac = 0$

$\Rightarrow \quad (8p - 40)^2 - 4 \times (9) \times (4p^2 + 60) = 0$

$\Rightarrow \quad 64p^2 - 640p + 1600 - 144p^2 - 2160 = 0$

$\Rightarrow \quad -80p^2 - 640p - 560 = 0$

$\div -80 \qquad \Rightarrow \quad p^2 + 8p + 7 = 0$

$(p + 7)(p + 1) = 0$

$\boxed{p = -7}$ $\qquad\qquad$ $\boxed{p = -1}$

$9x^2 \cancel{+9} - (8p - 40)x + 4p^2 + 60 = 0$

$\Rightarrow \quad 9x^2 + 96x + 256 = 0$

$\Rightarrow \quad (3x + 16)(3x + 16) = 0$

$x = -16/3$

$y = x + 4$

$= -4/3$

$(-16/3, -4/3)$

$\Rightarrow \quad 9x^2 + 48x + 64 = 0$

$(3x + 8)(3x + 8) = 0$

$x = -8/3$

$y = x + 4$

$= 4/3$

$(-8/3, 4/3)$