Further Pure 1 – January 2011

(1) a) $\alpha + \beta = 6 \qquad \alpha\beta = 18$

b) SUM $\quad \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 6^2 - 2(18) = 0$

PRODUCT $\quad \alpha^2\beta^2 = (\alpha\beta)^2 = 18^2 = 324$

$\Rightarrow x^2 - [\text{SUM}]x + [\text{PRODUCT}] = 0$

$\Rightarrow x^2 + 324 = 0$

c) $x^2 = -324$

$\Rightarrow x = \pm\sqrt{-324}$

$\Rightarrow x = \pm 18i$ which are α^2 & β^2 as they are roots

(2) a) $\int_p^q 2/x^3 = \int_p^q 2x^{-3} = \left[-x^{-2}\right]_p^q$

$= \left[\frac{-1}{x^2}\right]_p^q = \frac{-1}{q^2} - \frac{-1}{p^2}$

$= 1/p^2 - 1/q^2$

b) i) $\int_0^2 2/x^3 \, dx = \int_p^2 2/x^3 = 1/p^2 - 1/2^2$

$= 1/p^2 - 1/4$

As $p \to 0$, $1/p^2$ does not converge to a limit

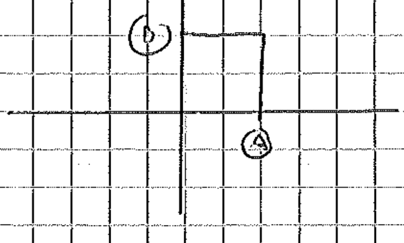
$\therefore \int$ has no value

ii) $\int_2^\infty 2/x^3 = \int_2^q 2/x^3 = 1/2^2 - 1/q^2$

$= 1/4 - 1/q^2$

As $q \to \infty$, $1/q^2 \to 0 \Rightarrow \therefore \int \to 1/4$

(3) a)



i) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

b) i) AB

$$\begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -20 & 14 \\ 14 & -10 \end{bmatrix}$$

ii) $A + B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$

$(A+B)^2 =$

$$\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} -25 & 0 \\ 0 & -25 \end{bmatrix} = -25 I$$

c) i) $\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = 5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ∴ Rotation 90° cw, enlargement SF 5

ii) $-25 I$ = Rotation 180°, enlargement SF 25

iii) $625 I$ = Enlargement SF 625

(4) $\theta = 2n\pi + \alpha$, $\theta = 2n\pi + (\pi - \alpha)$

Key angle $= \sin^{-1}(-1/2) = -\pi/6$

$4x - 2\pi/3 = 2n\pi - \pi/6$, $4x - 2\pi/3 = 2n\pi + 7\pi/6$

$4x = 2n\pi + \pi/2$ $4x = 2n\pi + 11\pi/6$

$x = 1/2\, n\pi + \pi/8$ $x = 1/2\, n\pi + 11\pi/24$

(5) a) i) $z^2 = (1/2 - i)(1/2 - i) = 1/4 - 1/2 i - 1/2 i + i^2$

$= 1/4 - i - 1 = -3/4 - i$

ii) $z^2 + z^* + 1/4$

$= (-3/4 - i) + (1/2 + i) + 1/4$

$= 0$ ∴ z is a root

b) $z_2^2 = (1/2 + i)(1/2 + i) = 1/4 + 1/2 i + 1/2 i + i^2$

$= 1/4 + i - 1 = -3/4 + i$

$\cdot\ z^2 + z^* + 1/4$

$= (-3/4 + i) + (1/2 - i) + 1/4 = 0$

$\therefore z_2$ is a root

c) z_1 & z_2 are the roots

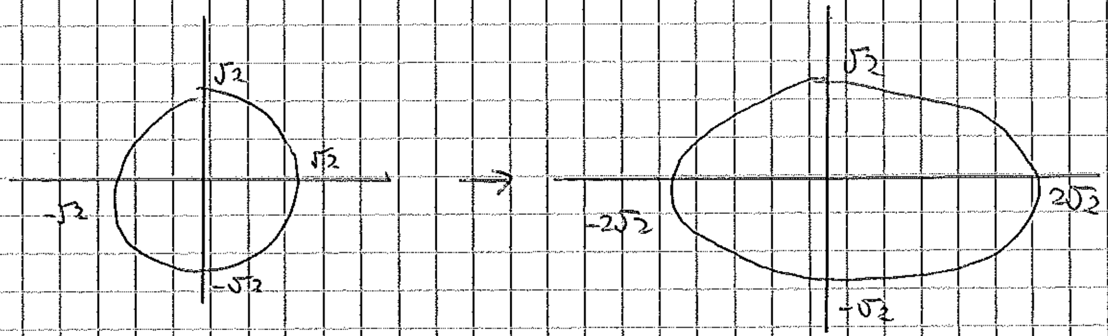
If they are real, then $z = z^*$ as no imaginary terms

$\Rightarrow z^2 + z + 1/4 = 0$

$\Rightarrow (z + 1/2)(z + 1/2) = 0$

$\Rightarrow z = -1/2$

(6) a)



b) i) $\frac{x^2}{2^2} + y^2 = 2 \quad \Rightarrow \quad \frac{x^2}{4} + y^2 = 2$

ii) Eq of L was $x + y = 2$

This was tangent at $(1, 1)$

Curve stretched SF 2 is x-direction $\Rightarrow (1, 1) \rightarrow (2, 1)$

$\therefore x$ has been replaced with $x/2$

$\Rightarrow (x/2) + y = 2$

(7) a) For vertical asymptote, $x^2 + 9$ must $= 0$
But, no real solutions for this

Horizontal: as $x \to \infty$, $y \to 0/1 \Rightarrow y = 0$

b) $y = k \Rightarrow k = \frac{x - 4}{x^2 + 9} \Rightarrow k(x^2 + 9) = x - 4$

$kx^2 + 9k = x - 4$

$kx^2 - x + (9k + 4) = 0$

c) For real roots, $b^2 - 4ac \geq 0$

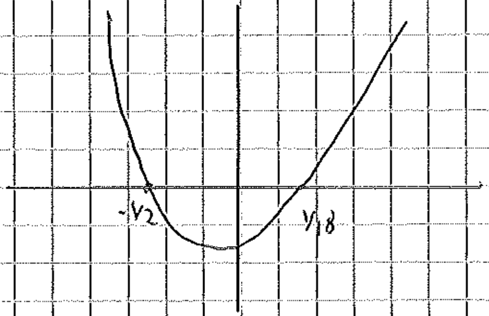
$\rightarrow (-1)^2 - 4(k)(9k+4) \geq 0$

$\rightarrow 1 - 36k^2 - 16k \geq 0$

$\rightarrow 36k^2 + 16k - 1 \leq 0$

$\rightarrow (18k - 1)(2k + 1) \leq 0$

$k = 1/18 \qquad k = -1/2$



graph below axes when

$-1/2 \leq k \leq 1/18$

d) Stationary if $b^2 - 4ac = 0$

ie $k = 1/18$ or $k = -1/2$

$\boxed{k = 1/18}$

$1/18\, x^2 - x + (9/18 + 4) = 0$

$1/18\, x^2 - x + 9/2 = 0$

$x^2 - 18x + 81 = 0$

$(x-9)(x-9) = 0$

$\rightarrow x = 9$

$y = k \rightarrow 1/18$ $\qquad (9, 1/18)$

$\boxed{k = -1/2}$

$-1/2\, x^2 - x + (-9/2 + 4) = 0$

$-1/2\, x^2 - x - 1/2 = 0$

$x^2 + 2x + 1 = 0$

$(x+1)(x+1) = 0$

$\rightarrow x = -1$

$y = k = -1/2$ $\qquad (-1, -1/2)$

(8) a) $x^3 + 2x^2 + x - 100,000 = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_1 = 50$

$f(50) = 50^3 + 2 \times 50^2 + 50 - 100,000 = 30,050$

$f'(x) = 3x^2 + 4x + 1$

$f'(50) = 3(50^2) + 4(50) + 1 = 7701$

$$\rightarrow x_2 = 50 - \frac{30,050}{7701} = 46.097...$$

b) i) $S_n = \sum r(3r+1) = 3\sum r^2 + \sum r$

$= \frac{1}{2} n(n+1)(2n+1) + \frac{1}{2} n(n+1)$

$= \frac{1}{2} n(n+1)[2n+1 + 1]$

$= \frac{1}{2} n(n+1)(2n+2)$

$= 2 \cdot \frac{1}{2} n(n+1)(n+1)$

$= n(n+1)^2$

ii) $n(n+1)^2 > 100,000$

$n(n+1)(n+1) > 100,000$

$n(n^2 + 2n + 1) > 100,000$

$n^3 + 2n^2 + n > 100,000$

$\rightarrow n^3 + 2n^2 + n - 100,000 > 0$

c) From a), $n = 46.097$ is a good approximation

$S_{46} = 46(47)^2 = 101\,614$

$S_{45} = 45(46)^2 = 95\,220$

$\therefore n = 46$ is the lowest