Further Pure 1 - May / June 2011

(1) $y_{n+1} = y_n + h\, f(x_n)$

$x_1 = 2$

$y_1 = 3$

$h = 0.5$

$y_2 = 3 + 0.5 \times 1/\sqrt{2+2}$

$= 3.25$

$x_2 = 2.5$

$y_2 = 3.25$

$h = 0.5$

$y_3 = 3.25 + 0.5 \times 1/\sqrt{2+2.5}$

$= 3.4857\ldots$

(2) a) $\alpha + \beta = -6/4 = -3/2$

$\alpha\beta = 3/4$

b) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (-3/2)^2 - 2(3/4) = 3/4$

c) $\boxed{\text{SUM}}$ $3\alpha - \beta + 3\beta - \alpha = 2\alpha + 2\beta$

$= 2(\alpha+\beta)$

$= 2(-3/2) = -3$

$\boxed{\text{PRODUCT}}$ $(3\alpha - \beta)(3\beta - \alpha) = 9\alpha\beta - 3\alpha^2 - 3\beta^2 + \alpha\beta$

$= 10\alpha\beta - 3(\alpha^2 + \beta^2)$

$= 10(3/4) - 3(3/4) = 21/4$

$x^2 - \boxed{\text{SUM}}\, x + \boxed{\text{PRODUCT}} = 0$

$\rightarrow x^2 + 3x + 21/4 = 0$

$\rightarrow 4x^2 + 12x + 21 = 0$

(3) a) $(z - i)(z^* - i)$

$= zz^* - zi - z^*i + i^2$

$= (x+iy)(x-iy) - (x+iy)i - (x-iy)i - 1$

$= x^2 - xyi + xyi - i^2y - xi - yi^2 - xi + yi^2 - 1$

$= x^2 + y^2 - 2xi - 1$

$\boxed{\text{REAL}}$ $x^2 + y^2 - 1$

$\boxed{\text{IMAG}}$ $-2x$

b) $\boxed{\text{REAL}}$ $x^2 + y^2 - 1 = 24$

$\boxed{\text{IMAG}}$ $-2x = -8 \Rightarrow x = 4$

$\therefore 4^2 + y^2 - 1 = 24$

$y^2 = 9 \Rightarrow y = \pm 3$

$\therefore z = (4 + 3i)$ or $z = (4 - 3i)$

(14) a) $y = ab^x$

$\log_{10}(y) = \log_{10}(ab^x)$

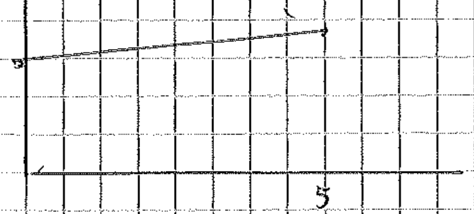
$\log_{10}(y) = \log_{10}(a) + x\log_{10}(b)$

$Y = c + mx$

$c = \log_{10}(a)$ $\qquad$ $m = \log_{10}(b)$

$\Rightarrow a = 10^c$ $\qquad$ $\Rightarrow b = 10^m$

b)



$x = 1,\ y = 12 \Rightarrow Y = 1.07918\ldots$

$x = 5,\ y = 27 \Rightarrow Y = 1.43136\ldots$

c) i) $x = 3,\ Y = 1.25 \Rightarrow y = 10^{1.25} = 17.78\ldots$

$= 18$ (2sf)

ii) $a = 10^c = 10^1 = 10$ (y int = 1)

(5) a) $\theta = 2n\pi \pm a$

Key angle $= \cos^{-1}(\sqrt{3}/2) = \pi/6$

$\Rightarrow 3x - \pi/6 = 2n\pi \pm \pi/6$

$\Rightarrow 3x = 2n\pi + \pi/6 \pm \pi/6$

$\Rightarrow x = \frac{2}{3}n\pi + \pi/18 \pm \pi/18$

b) Try $n = 7 \Rightarrow x = \frac{43}{14}\pi$ (too small)

Try $n = 8 \Rightarrow x = \frac{16}{3}\pi$, taking -ve

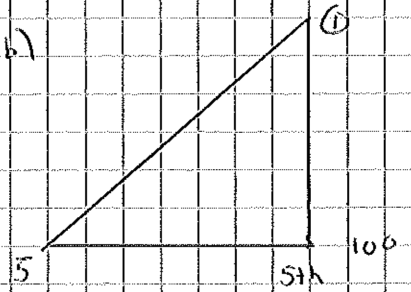
$\approx 16.755...$

$\therefore n = 8$

(6) a) $5^3 + 3(5^2)h + 3(5)(h^2) + h^3$

$= 125 + 75h + 15h^2 + h^3$

b)



(1) $= (5+h)^3 - (5+h)^2$

$= h^3 + 15h^2 + 75h + 125 - (h^2 + 10h + 25)$

$= h^3 + 14h^2 + 65h + 100$

Gradient $= \dfrac{h^3 + 14h^2 + 65h + 100 - 100}{h}$

$= h^2 + 14h + 65$

i) As $h \to 0$, gradient $\to 65$

This is the gradient of the curve.

(7) a) i) A^2

$$\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}$$

ii) A^3

$$\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8I$$

b) i) $8I$ = Enlargement, scale factor 8, centre (0,0)

ii) $A = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ SF must be $\sqrt[3]{8} = 2$

$$\Rightarrow 2\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$= 2 \begin{bmatrix} -\cos(60) & -\sin(60) \\ \sin(60) & -\cos(60) \end{bmatrix}$$

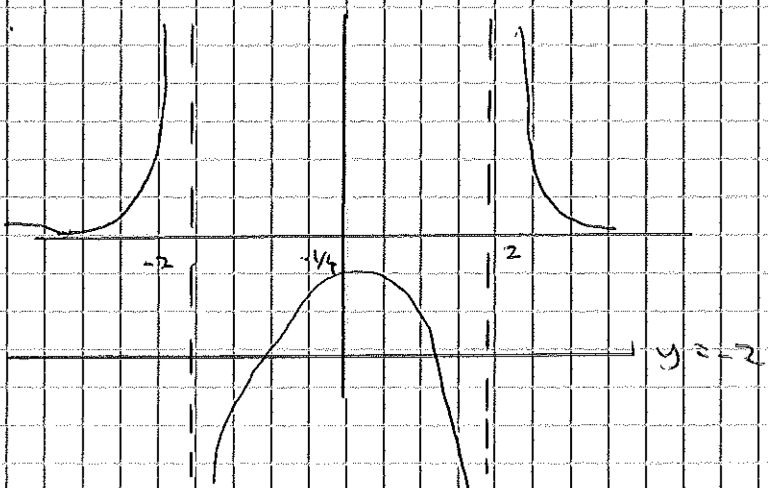
$$= 2 \begin{bmatrix} \cos(120) & -\sin(120) \\ \sin(120) & \cos(120) \end{bmatrix}$$

= Enlargement, SF 2
Rotation ~~[illegible]~~ 120° anti-clockwise.

(8) a) i) Denominator : $x = 2$, $x = -2$

numerator : as $x \to \infty$, $y \to 0/1 \Rightarrow y = 0$

ii(b) :



$x = 0$, $y = -1/4$

b) Need crossing points:

$$\frac{1}{x^2-4} = -2$$

$$\rightarrow \quad 1 = -2x^2 + 8$$

$$2x^2 = 7$$

$$x^2 = 7/2$$

$$x = \pm\sqrt{7/2}$$

curve <u>below</u> line : $-2 < x < -\sqrt{7/2}$

and $\sqrt{7/2} < x < 2$

(9) a) i) As it's a reflection in $y = x$, it crosses $y = x$ at A

$\rightarrow \quad x = \frac{1}{8}x^2$

$8x = x^2$

$x^2 - 8x = 0$

$x(x - 8) = 0$

$x = 0$ Origin

$x = 8 = $ Point A

$y = 8 \rightarrow (8, 8)$

ii) x, y swap $\rightarrow \quad x = \frac{1}{8}y^2$

iii) Symmetry means solutions lie on perpendicular line

$\rightarrow$ gradient of -1 $\quad (y = -x + c)$

b) $y = -x + c \quad \rightarrow \quad -x + c = \frac{1}{8}x^2$

$-8x + 8c = x^2$

$x^2 + 8x - 8c = 0$

For 2 crossing points, $b^2 - 4ac > 0$

$\rightarrow \quad 8^2 - 4(1)(-8c) > 0$

$64 + 32c > 0$

$32c > -64$

$\rightarrow \quad c > -2 \quad (-64/32)$

ii) When L touches curve, $b^2 - 4ac = 0 \quad \rightarrow \quad c = -2$

$\rightarrow \quad y = -x - 2$

Sub into (P) $\rightarrow \quad -x - 2 = \frac{1}{8}x^2$

$-8x - 8 = x^2$

$x^2 + 8x + 16 = 0$

$\rightarrow (x + 4)(x + 4) = 0$

$\rightarrow \quad x = -4, \quad y = -(-4) - 2 = 2 \quad \rightarrow (-4, 2)$

Other co-ordinates are reflection in $y = x \quad \rightarrow \quad (4, -2)$