Further Pure 1 - June 2004

① a) $\alpha + \beta = -\frac{1}{2}$ $\quad\quad$ $\alpha\beta = -4$

b) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-\frac{1}{2})^2 - 2(-4) = 33/4$

c) $\boxed{\text{SUM}}$ $4\alpha^2 + 4\beta^2 = 4(\alpha^2 + \beta^2)$

$= 4(33/4) = 33$

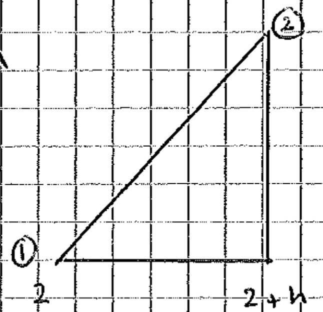
$\boxed{\text{PRODUCT}}$ $4\alpha^2 \times 4\beta^2 = 16\alpha^2\beta^2 = 16(\alpha\beta)^2$

$= 16(-4)^2 = 256$

$x^2 - \boxed{\text{SUM}}\,x + \boxed{\text{PRODUCT}} = 0$

$\Rightarrow x^2 - 33x + 256 = 0$

② a)



①: $(2)^2 - 6(2) + 5 = -3$

②: $(2+h)^2 - 6(2+h) + 5$

$= 4 + 4h + h^2 - 12 - 6h + 5$

$= h^2 - 2h - 3$

$\therefore$ Grad $= \dfrac{h^2 - 2h - 3 - (-3)}{h}$

$= h - 2$

b) As $h \to 0$, grad $\to -2$

③ a) i) $z^2 = (x + 2i)(x + 2i)$

$= x^2 + 4xi - 4$ $\leftarrow$ $4i^2$

$\boxed{\text{REAL}}$ $x^2 - 4$

$\boxed{\text{IMAG}}$ $4x$

ii) $z^2 + 2z^*$

$= x^2 + 4xi - 4 + 2(x - 2i)$

$= x^2 - 4 + 4xi + 2x - 4i$

$= x^2 + 2x - 4 + 4xi - 4i$

$\boxed{\text{REAL}}$ $x^2 + 2x - 4$

$\boxed{\text{IMAG}}$ $4x - 4$

b) For real, $4x - 4 = 0$ ie $\boxed{\text{IMAG}} = 0$

$\rightarrow x = 1$

(4) a) $y = ab^x$

$\log_{10}(y) = \log_{10}(ab^x)$

$\log_{10}(y) = \log_{10}(a) + x\log_{10}(b)$

$\rightarrow Y = \underset{\uparrow\ \log_{10}(b)}{m}x + \underset{\uparrow\ \log_{10}(a)}{c}$

b) i) $x = 2.3 \rightarrow Y \approx 1.1$

$\rightarrow \log_{10}(y) = 1.1$

$\rightarrow y = 10^{1.1} = 12.589... = 12.6$ (1dp)

ii) $y = 80 \rightarrow Y = \log_{10}(80)$

$\rightarrow Y = 1.903$

$\rightarrow x \approx 1.1$

(5) a) For cos: $\theta = 2n\pi \pm \alpha$

Key angle $(\alpha) = \cos^{-1}(1/2) = \pi/3$

$\rightarrow 3x - \pi = 2n\pi \pm \pi/3$

$\rightarrow 3x = 2n\pi + \pi \pm \pi/3$

$\rightarrow x = 2/3\, n\pi + \pi/3 \pm \pi/9$

b) Try $n = 1 \rightarrow x = 2/3\,\pi + \pi/3 \pm \pi/9$ = Too small

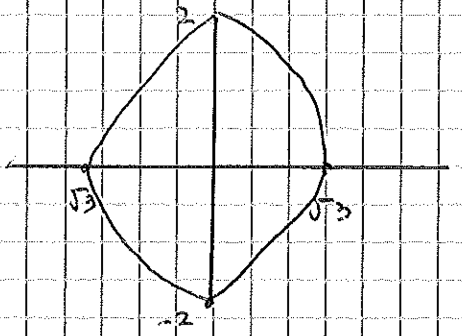
Try $n = 15 \rightarrow x = 31\pi/3 \pm \pi/9 = 10\,4/9\,\pi$ $(94\pi/9)$

$\rightarrow 10\,2/9\,\pi$ $(92\pi/9)$

Try $n = 16 \rightarrow x = 33\pi/3 \pm \pi/9 = 10\,8/9\,\pi$ $(98\pi/9)$

$\rightarrow$ Too big

6) a)



x-axis : $(\pm\sqrt{3}, 0)$

y-axis : $(0, \pm 2)$

b) $\frac{x^2}{3} + \frac{(y/2)^2}{4} = 1$

$\rightarrow \frac{x^2}{3} + \frac{y^2}{16} = 1$

c) $4x^2 - 8x + 3y^2 + 6y = 5$

$4[x^2 - 2x] + 3[y^2 + 2y] = 5$

$4[(x-1)^2 - 1] + 3[(y+1)^2 - 1] = 5$

$4(x-1)^2 - 4 + 3(y+1)^2 - 3 = 5$

$4(x-1)^2 + 3(y+1)^2 = 12$

$\div 12$ $\quad \frac{(x-1)^2}{3} + \frac{(y+1)^2}{4} = 1$

$\rightarrow$ Translation : $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

7) a) i) $\begin{bmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

ii) $y = 1/\sqrt{3}\, x \qquad \tan^{-1}(1/\sqrt{3}) = 30$

$\rightarrow y = [\tan(30)]\, x$

$\rightarrow \begin{bmatrix} \cos(60) & \sin(60) \\ \sin(60) & -\cos(60) \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} = 2 \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = 2 \begin{bmatrix} \text{Part a) ii} \end{bmatrix}$

= Enlargement, SF 2, centre (0,0)
& reflection in line $y = [\tan 30]\, x$

c) A followed by B = BA

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} = 4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

= Enlargement SF 4, centre $(0,0)$

+ Reflection in line $y = x$

(8) a) Denominator $= 0$ at $x = 1$ + $x = 5$

As $x \to \infty$, $y \to 1/1 \to y = 1$

b) $y = -1 \to -1 = \frac{x^2}{(x-1)(x-5)}$

$\to -(x-1)(x-5) = x^2$

$\to -x^2 + 6x - 5 = x^2$

$0 = 2x^2 - 6x + 5$

check discriminant: $b^2 - 4ac$

$\to (-6)^2 - 4 \times 2 \times 5 = -4$

negative discriminant, $\therefore$ no real solutions

$\therefore$ no points of intersection

c) i) $y = k \to k = \frac{x^2}{(x-1)(x-5)}$

$\to k[x^2 - 6x + 5] = x^2$

$\to kx^2 - 6kx + 5k = x^2$

$\to kx^2 - x^2 - 6kx + 5k = 0$

$\to (k-1)x^2 - 6kx + 5k = 0$

ii) For equal roots, $b^2 - 4ac = 0$

$$\Rightarrow (-6k)^2 - 4 \times (k-1) \times 5k = 0$$

$$\Rightarrow 36k^2 - 20k(k-1) = 0$$

$$36k^2 - 20k^2 + 20k = 0$$

$$16k^2 + 20k = 0$$

$$4k^2 + 5k = 0$$

$$k(4k+5) = 0$$

d) For stationary points, $b^2 - 4ac = 0$

$$k(4k+5) = 0$$

$k = 0$

$$0 = \frac{x^2}{(x-1)(x-5)}$$

$\Rightarrow x = 0$

$\Rightarrow y = 0$

$\Rightarrow (0, 0)$

$k = -5/4$

$$-5/4 = \frac{x^2}{(x-1)(x-5)}$$

$$-5/4\,(x-1)(x-5) = x^2$$

$$-5/4\left[x^2 - 6x + 5\right] = x^2$$

$$-5\left[x^2 - 6x + 5\right] = 4x^2$$

$$-5x^2 + 30x - 25 = 4x^2$$

$$0 = 9x^2 - 30x + 25$$

$$0 = (3x-5)(3x-5)$$

$$x = 5/3$$

$$\Rightarrow y = \frac{(5/3)^2}{(5/3-1)(5/3-5)} = -5/4$$

$$\Rightarrow (5/3, -5/4)$$