Further Pure 1 – January 2008

① $z = 2 + i \qquad z^* = 2 - i$

$x + 3iy = 2 + i + 4i(2 - i)$

$= 2 + i + 8i + 4$

$x + 3iy = 6 + 9i$

Real: $x = 6$

Imaginary: $3y = 9 \Rightarrow y = 3$

② $dy/dx = 2^x \qquad y_{n+1} = y_n + h f(x_n)$

$f(x) = 2^x$
$x_1 = 1$
$y_1 = 4$
$h = 0.01$

$y_2 = 4 + 0.01 \times 2^1$
$= 4.02$

$x_2 = 1.01$
$y_2 = 4.02$
$h = 0.01$

$y_3 = 4.02 + 0.01 \times 2^{1.01}$
$= 4.04014$

③ Radians: $\theta = n\pi + a$

Key angle: $\tan^{-1}(1) = \pi/4 = a$

So: $4(x - \pi/8) = n\pi + \pi/4$

$x - \pi/8 = n\pi/4 + \pi/16$

$x = n\pi/4 + 3\pi/16$

④ a) Key formula: $r = \frac{1}{2}n(n+1) \qquad r^3 = \frac{1}{4}n^2(n+1)^2$

$\sum r^3 - 6r$

$= \sum r^3 - 6\sum r$

$= \frac{1}{4}n^2(n+1)^2 - 3n(n+1)$

$= \frac{1}{4}n(n+1)\left[n(n+1) - 12\right]$

$= \frac{1}{4}n(n+1)\left[n^2 + n - 12\right]$

$= \frac{1}{4}n(n+1)(n+4)(n-3)$

b) If $n = 1000$

$\rightarrow S = 250(1001)(1004)(997)$

$2008 = 251 \times 2^3$

Need to show that 251 and $2 \times 2 \times 2$ are factors of S

$1004 = 251 \times 4 = 251 \times 2 \times 2$

$250 = 125 \times 2$

$\therefore$ Factors of 2008 ~~and~~ are in S, so S is a multiple of 2008.

(5) a) Equation is: $\dfrac{x^2}{2^2} - \dfrac{y^2}{1^2} = 1$

Asymptotes at $y = {}^{b}/_{a}x$, $y = -{}^{b}/_{a}x$

so $y = {}^{1}/_{2}x$ and $y = -{}^{1}/_{2}x$

b) $x = 4 \Rightarrow {}^{16}/_{4} - y^2 = 1$

$4 - y^2 = 1$

$\rightarrow y^2 = 3 \rightarrow y = \pm\sqrt{3}$

c) i) Translation $\uparrow 2 \rightarrow 2 + \sqrt{3}$ and $2 - \sqrt{3}$

ii) Equation: $\dfrac{x^2}{4} - (y-2)^2 = 1$

Asymptotes: $y = {}^{1}/_{2}x + 2$ and $y = -{}^{1}/_{2}x - 2$

(6) a) i) $\begin{pmatrix} \sqrt{3} & 3 \\ 3 & -\sqrt{3} \end{pmatrix}$

$\begin{pmatrix} \sqrt{3} & 3 \\ 3 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} = 12I$

ii) $M = \begin{pmatrix} \sqrt{3} & 3 \\ 3 & -\sqrt{3} \end{pmatrix}$ Need: $\begin{pmatrix} {}^{1}/_{2} & {}^{\sqrt{3}}/_{2} \\ {}^{\sqrt{3}}/_{2} & -{}^{1}/_{2} \end{pmatrix}$

So, take out factor of $2\sqrt{3} \rightarrow 2\sqrt{3}\begin{pmatrix} {}^{1}/_{2} & {}^{\sqrt{3}}/_{2} \\ {}^{\sqrt{3}}/_{2} & -{}^{1}/_{2} \end{pmatrix}$

$\rightarrow q = 2\sqrt{3}$

b) i) SF $= 2\sqrt{3}$ or $\sqrt{12}$

(ii) Reflections look like: $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$

$\therefore \theta = 30$

$\Rightarrow$ Reflection in $y = \tan(30)x$

$\rightarrow y = \sqrt{3}/3 \, x$

c) $M^4 = M^2 \times M^2$

$= 12I \times 12I = 144I$

= Enlargement, scale factor 144.

(7) a) i) $y = x^3 - x + 1$ $x = (-1 + h)$

$\Rightarrow y = (-1+h)^3 - (-1+h) + 1$

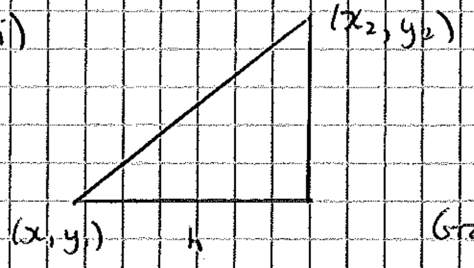
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow y = (-1)^3 + 3(-1)^2h + 3(-1)h^2 + h^3 + 1 - h + 1$

$y = -1 + 3h + 3h^2 + h^3 + 1 - h + 1$

$\Rightarrow y = 1 + 2h - 3h^2 + h^3$

ii)



$x_1 = -1$ $y_1 = (-1)^3 - (-1) + 1 = 1$

$x_2 = -1 + h$ $y_2 = 1 + 2h - 3h^2 + h^3$

Gradient $= \dfrac{y_2 - y_1}{h}$

$= \dfrac{1 + 2h - 3h^2 + h^3 - 1}{h}$

$= \dfrac{2h - 3h^2 + h^3}{h} = 2 - 3h + h^2$

iii) As $h \to 0$, gradient $\to 2 - 3(0) + (0)^2 = 2$

gradient of chord $\to$ gradient of tangent

b) i) $x^3 - x + 1 = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

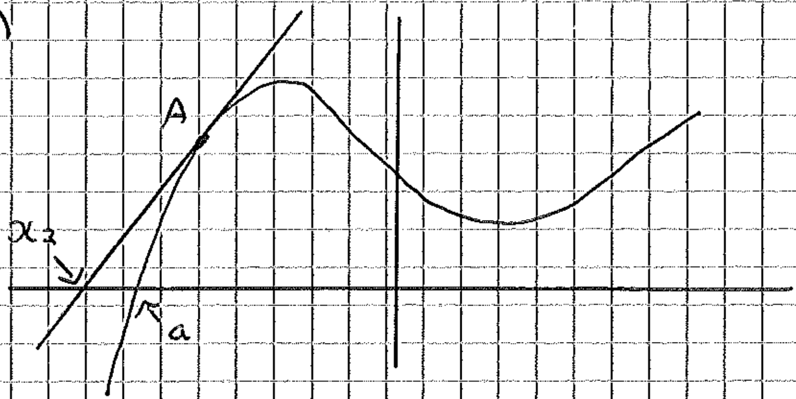
$f(x) = x^3 - x + 1$

$f'(x) = 3x^2 - 1$ $\qquad x_2 = -1 - \frac{(-1)^3 - (-1) + 1}{3(-1)^2 - 1}$

$x_1 = -1$

$\qquad = -1 - 1/2 = -3/2$

ii)



(8) a) i) $x^2 - 2x + 4 = 0$

$\alpha + \beta = 2$

$\alpha\beta = 4$

$\boxed{\text{SUM}}$ $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= 2^3 - 3 \times 4 \times 2 = -16$

$\boxed{\text{PRODUCT}}$ $\alpha^3\beta^3 = (\alpha\beta)^3$

$= 4^3 = 64$

$x^2 - \boxed{\text{SUM}}x + \boxed{\text{PRODUCT}} = 0$

$\rightarrow x^2 + 16x + 64 = 0$

ii) Discriminant: $b^2 - 4ac$

$= 16^2 - 4 \times 1 \times 64 = 0$

$\therefore$ Roots are real and equal

b) Use formula: $\frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 4}}{2}$

$= \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{-3}}{2}$

$= \dfrac{1 \pm \sqrt{-3}}{} \qquad = 1 \pm i\sqrt{3}$

c) From b) α and β = $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$

From a) α^3 and β^3 are equal

Therefore $(1 + i\sqrt{3})^3 = (1 - i\sqrt{3})^3$

(9) a) $y = \dfrac{2}{x(x-4)}$

Asymptote : $x = 0$ and $x = 4$

As $x \to \infty$, $y \to 2/\infty \to 0$

$\therefore$ Asymptote at $y = 0$

b) Let $y = k$

$$k = \frac{2}{x(x-4)}$$

$$kx(x-4) = 2$$

$$kx^2 - 4kx = 2$$

$$kx^2 - 4kx - 2 = 0$$

At stationary points, roots are equal, so $b^2 - 4ac = 0$

$$(-4k)^2 - 4 \times k \times (-2) = 0$$

$$16k^2 + 8k = 0$$

$$2k^2 + k = 0$$

$$k(2k + 1) = 0$$

$k = 0 \qquad k = -1/2$

$\boxed{k = 0}$ $\quad 0 = \dfrac{2}{(x)(x-4)} \to$ NO SOLUTION

$\boxed{k = -1/2}$ $\quad -1/2 = \dfrac{2}{x(x-4)} \to -1 = \dfrac{4}{x(x-4)}$

$\to -x(x-4) = 4$

$\to -x^2 + 4x = 4$

$\to x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)(x-2) = 0$

$\Rightarrow x = 2$ repeated solution ✓

$y = \frac{2}{2x-2} = -1/2$

$\therefore$ co-ordinates $= (2, -1/2)$

c)