Further Pure 1 - June 2006

① a) $\alpha + \beta = -\,{}^{-6}/_{3} = 2 \qquad \alpha\beta = {}^{2}/_{3}$

b) i) using Pascal: $\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

ii) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= 2^3 - 3\left({}^{2}/_{3}\right)(2)$

$= 8 - 4 = 4$

c) $\boxed{SUM} = 4$

$\boxed{PRODUCT} = \alpha^3 \times \beta^3 = (\alpha\beta)^3 = \left({}^{2}/_{3}\right)^3 = {}^{8}/_{27}$

$\Rightarrow x^2 - \boxed{SUM}\,x + \boxed{PRODUCT} = 0$

$\Rightarrow x^2 - 4x + {}^{8}/_{27} = 0$

$\Rightarrow 27x^2 - 108x + 8 = 0$

② $y_{n+1} = y_n + h f(x_n)$

$x_1 = 2$

$y_1 = 3$

$h = 0.2$

$f(2) = \log_{10}(2)$

$y_2 = 3 + 0.2 \times \log_{10}(2)$

$= 3.060205...$

$x_2 = 2.2$

$y_2 = 3.060520...$

$h = 0.2$

$f(2.2) = \log_{10}(2.2)$

$y_3 = 3.0605... + 0.2 \times \log_{10}(2.2)$

$= 3.128690...$

$= 3.129$ (3dp)

③ $\sum_{1}^{n} r^2 - r = \sum r^2 - \sum r$

$= {}^{1}/_{6}\, n(n+1)(2n+1) - {}^{1}/_{2}\, n(n+1)$

$= {}^{1}/_{6}\, n(n+1)\left[(2n+1) - 3\right]$

$= {}^{1}/_{6}\, n(n+1)\left[2n - 2\right]$

Take out Factor of 2!

$= {}^{1}/_{3}\, n(n+1)(n-1)$

④ GS: $\theta = 2n\pi \pm \alpha$

Key angle (α) $= \cos^{-1}(\sqrt{3}/2) = \pi/6$

$\Rightarrow 3x = 2n\pi \pm \pi/6$

$\Rightarrow x = \frac{2}{3}n\pi \pm \pi/18$

⑤ a) i) M^2

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

ii) $M^4 = M^2 \times M^2$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

b) In the form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\quad \theta = 45^\circ$

Direction = clockwise

$\Rightarrow$ Rotation 45° clockwise about $(0,0)$

c) $M^{2006} = M^{2000} \times M^6$

$= [M^8]^{250} \times M^6$

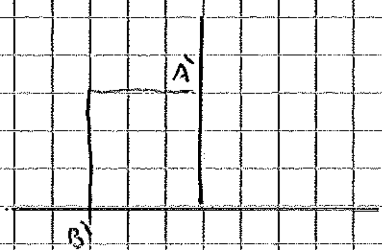
M^8 must $= I$ as it is 8 45° rotations

$= I \times M^6$

$= M^6$ = rotation 270° clockwise

or 90° anti-clockwise

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



⑥ a) $(z+i)^* = (x+iy+i)^*$

$= x - iy - i$

b) $x - iy - i = 2i(x+iy) + 1$

$= 2xi - 2y + 1$

$\boxed{\text{REAL}}$ $x = -2y + 1$ ①

$\boxed{\text{IMAG}}$ $-y - 1 = 2x$ ②

① $\times 2 \rightarrow 2x = -4y + 2$

sub ① into ② $\rightarrow -y - 1 = -4y + 2$

$\rightarrow 3y = 3 \rightarrow y = 1$

From ① $x = -2y + 1 \rightarrow x = -1$

$\rightarrow z = -1 + i$

⑦ a) y has been replaced by $(2y)$ [to give $4y^2$]

$\rightarrow$ stretch, scale factor $1/2$ parallel to y-axis

b) $x^2 - y^2 - 4x + 3 = 0$

$\rightarrow x^2 - 4x - y^2 + 3 = 0$

$\rightarrow (x-2)^2 - 4 - y^2 + 3 = 0$

$\rightarrow (x-2)^2 - y^2 = 1$

$\rightarrow$ Translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ or 2 units right in x direction

⑧ a) i) $f(1+h) = (1+h)^3 + (1+h)^2 - 1$

$= 1 + 3h + 3h^2 + h^3 + 1 + 2h + h^2 - 1$

$= h^3 + 4h^2 + 5h + 1$

$f(1) = 1 + 1 - 1 = 1$

$\therefore f(1+h) - f(1) = h^3 + 4h^2 + 5h + 1 - 1$

$= h^3 + 4h^2 + 5h$

ii) Gradient $= \dfrac{h^3 + 4h^2 + 5h}{h} = h^2 + 4h + 5$

As $h \rightarrow 0$, gradient $\rightarrow 5$

b) i) At intersection : $x + 1 = 1/x^2$

$\rightarrow x^3 + x^2 = 1$

$\rightarrow x^3 + x^2 - 1 = 0 = f(x)$

ii) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 1$

$f(x) = x^3 + x^2 - 1$

$f'(x) = 3x^2 + 2x$ $\qquad x_2 = 1 - \dfrac{1}{5}$

$f(1) = 1 + 1 - 1 = 1$ $\qquad = 4/5$

$f'(1) = 3 + 2 = 5$

c) $\displaystyle\int_1^{\infty} 1/x^2 = \int_1^{m} x^{-2} = \left[-x^{-1}\right]_1^{m}$

$= -1/m - (-1/1) = 1 - 1/m$

As $m \rightarrow \infty$, $1/m \rightarrow 0$

$\int \rightarrow 1$ = Area of Region

(9) a) i) Intersects x-axis when $y = 0$

$\rightarrow 0 = \dfrac{(x+1)(x-3)}{(x)(x-2)}$

$\rightarrow 0 = (x+1)(x-3) \rightarrow x = -1$ or $x = 3$

$\rightarrow (-1, 0)$ or $(3, 0)$

ii) A Denominator $= 0$ when $x = 0$ and $x = 2$

As $x \rightarrow \infty$, $y \rightarrow x^2/x^2 \rightarrow 1$

$\therefore$ Asymptotes : $x = 0$, $x = 2$, $y = 1$

b) i) $K = \dfrac{(x+1)(x-3)}{x(x-2)}$

$\rightarrow Kx(x-2) = (x+1)(x-3)$

$\rightarrow Kx^2 - 2Kx = x^2 - 2x - 3$

$\rightarrow Kx^2 - x^2 - 2Kx + 2x + 3 = 0$

$\rightarrow (K-1)x^2 + (2K-2)x + 3 = 0$

For intersection, $b^2 - 4ac \geqslant 0$

$\Rightarrow (2k-2)^2 - 4(k-1) \times 3 \geqslant 0$

$\Rightarrow 4k^2 - 8k + 4 - 12k + 12 \geqslant 0$

$\Rightarrow 4k^2 - 20k + 16 \geqslant 0$

$\Rightarrow k^2 - 5k + 4 \geqslant 0$

$\Rightarrow (k-1)(k-4) \geqslant 0$

ii) At st point, $b^2 - 4ac = 0$

$\Rightarrow (k-1)(k-4) = 0$

$\Rightarrow k = 1$ or $k = 4$

$(1-1)x^2 - (2-2)x + 3 = 0$

$\Rightarrow 3 = 0$

No solution

$(4-1)x^2 - (8-2)x + 3 = 0$

$\Rightarrow 3x^2 - 6x + 3 = 0$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)(x-1) = 0$

$x = 1, y = 4$

$= (1, 4)$

c)