FP1 – February 2005

① $x^2 - 5x - 2$

a) $\alpha + \beta = 5 \qquad \alpha\beta = -2$

b) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$

$= 5(-2) = -10$

c) $\boxed{\text{SUM}} = -10$

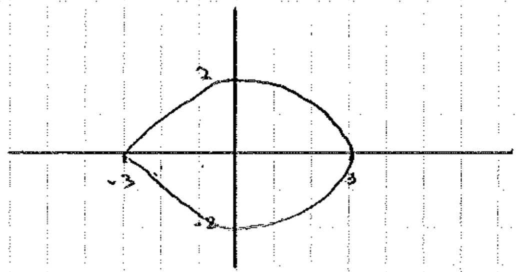
$\boxed{\text{PRODUCT}} \quad \alpha^2\beta \times \alpha\beta^2 = \alpha^3\beta^3 = (\alpha\beta)^3$

$= (-2)^3 = -8$

$\Rightarrow x^2 - \boxed{\text{SUM}}\,x + \text{PRODUCT} = 0$

$\Rightarrow x^2 + 10x - 8 = 0$

② a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \Rightarrow \dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$



b) $x = 1$

$\Rightarrow \dfrac{1}{9} + \dfrac{y^2}{4} = 1$

$1 + \dfrac{9y^2}{4} = 9$

$4 + 9y^2 = 36$

$9y^2 = 32$

$y^2 = \dfrac{32}{9} \Rightarrow y = \pm\sqrt{\dfrac{32}{9}}$

$\Rightarrow y = \dfrac{\pm\sqrt{32}}{3} = \pm\dfrac{4}{3}\sqrt{2}$

c) $\xrightarrow{1} \; = \dfrac{(x-1)^2}{9} + \dfrac{y^2}{4} = 1$

③ a) $z^* = x - iy$

b) $2z - iz^* = 2(x + iy) - i(x - iy)$

$= 2x + 2\cancel{x}yi - xi - y$ $\swarrow$ $+i^2y$

REAL : $2x - y$

IMAG : $-x + 2y$

c) REAL : $2x - y = 0$ ①

IMAG : $-x + 2y = 3$ ②

① × 2 $\Rightarrow$ $4x - 2y = 0$ (+)

$-x + 2y = 3$

$3x = 3 \Rightarrow x = 1$

① $2x - y = 0$

$\Rightarrow 2 - y = 0 \Rightarrow y = 2$

$z = x + iy \Rightarrow z = 1 + 2i$

④ a) $\int_2^{\infty} 8x^{-3}\,dx \Rightarrow \int_2^{K} 8x^{-3} \Rightarrow \left[-4x^{-2}\right]_2^K$

$\Rightarrow -4(K^{-2}) - (-4)(2)^{-2}$

$\Rightarrow 1 - \frac{4}{K^2}$

As $K \to \infty$, $\frac{4}{K^2} \to 0$, $\int \to 1 - 0 = 1$ $\therefore$ Integral has value 1

b) From a) $\Rightarrow \left[-4x^{-2} + x\right]_2^K$

$\Rightarrow -4(K)^{-2} + K - \left((-4)2^{-2} + 2\right)$

$\Rightarrow \frac{-4}{K^2} + K + 1 - 2 \Rightarrow -1 - \frac{4}{K^2} + K$

As $K \to \infty$, the $\int \to$ no value as $K \not\to 0$

c) $\int_2^K 8x^{-3}(x+1)\,dx$

$\Rightarrow \int_2^K 8x^{-2} + 8x^{-3}\,dx \Rightarrow \left[-8x^{-1} - 4x^{-2}\right]_2^K$

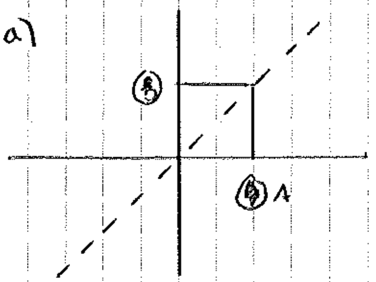
$= \frac{-8}{K} - \frac{4}{K^2} - \left(\frac{(-8)}{2} - \frac{4}{2^2}\right)$

$= \frac{-8}{K} - \frac{4}{K^2} + 4 + 1$

As $K \to \infty$, $\int \to 5$

$\frac{-8}{K} \to 0$, $\frac{4}{K^2} \to 0$

(5) a)



Reflection in line $y = x$

b) Anti-clockwise rotation $= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ $\theta = 60°$

$= \begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

c) $(T_2)\ (T_1)\ (\text{object})$

$T_2 \times T_1$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

(6) a) General solution for cos: $\theta = 2n\pi \pm a$

Key angle: $\cos^{-1}(1/\sqrt{2}) = \pi/4$

$2x + \pi/6 = 2n\pi \pm \pi/4$

$2x = 2n\pi \pm \pi/4 - \pi/6$

$\rightarrow x = n\pi \pm \pi/8 - \pi/12$

b) Either: realise $\cos(2x)$ has twice as many solutions

Or: Try values of n

$n = 0 \rightarrow x = \pi/8 - \pi/12 = \pi/24$

$n = 1 \rightarrow x = \pi + \pi/8 - \pi/12 = 25\pi/24$

$n = 1 \rightarrow x = \pi - \pi/8 - \pi/12 = 19\pi/24$

$n = 2 \rightarrow x = 2\pi - \pi/8 - \pi/12 = 43\pi/24$

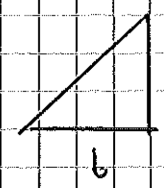
$= 4$ roots

(7) a)

X	2.25	16	25	42.25	64
Y	125	250	343	512	729

See graph

b) a = gradient



$= 60/6 = 10$

b = y intercept $= 100$

(8) a) $f(x) = x^3 - 2x - 1$

$f'(x) = 3x^2 - 2$

$x_1 = 1$

$f(1) = 1^3 - 2(1) - 1 = -2$

$f'(1) = 3(1)^2 - 2 = 1$

$\therefore x_2 = 1 - {-2}/{1} = 3$

b) See sheet

c) $f(2) = 2^3 - 2(2) - 1 = 3$

$3 > 0$, so the root must be less than 2

$\therefore x = 1$ is a better approximation than $x = 3$

d) $x_1 = 1.6$

$f(1.6) = 1.6^3 - 2(1.6) - 1 = -0.104$

$f'(1.6) = 3(1.6)^2 - 2 = 5.68$

$x_2 = 1.6 - {-0.104}/{5.68}$

$= 1.618$ (3dp)

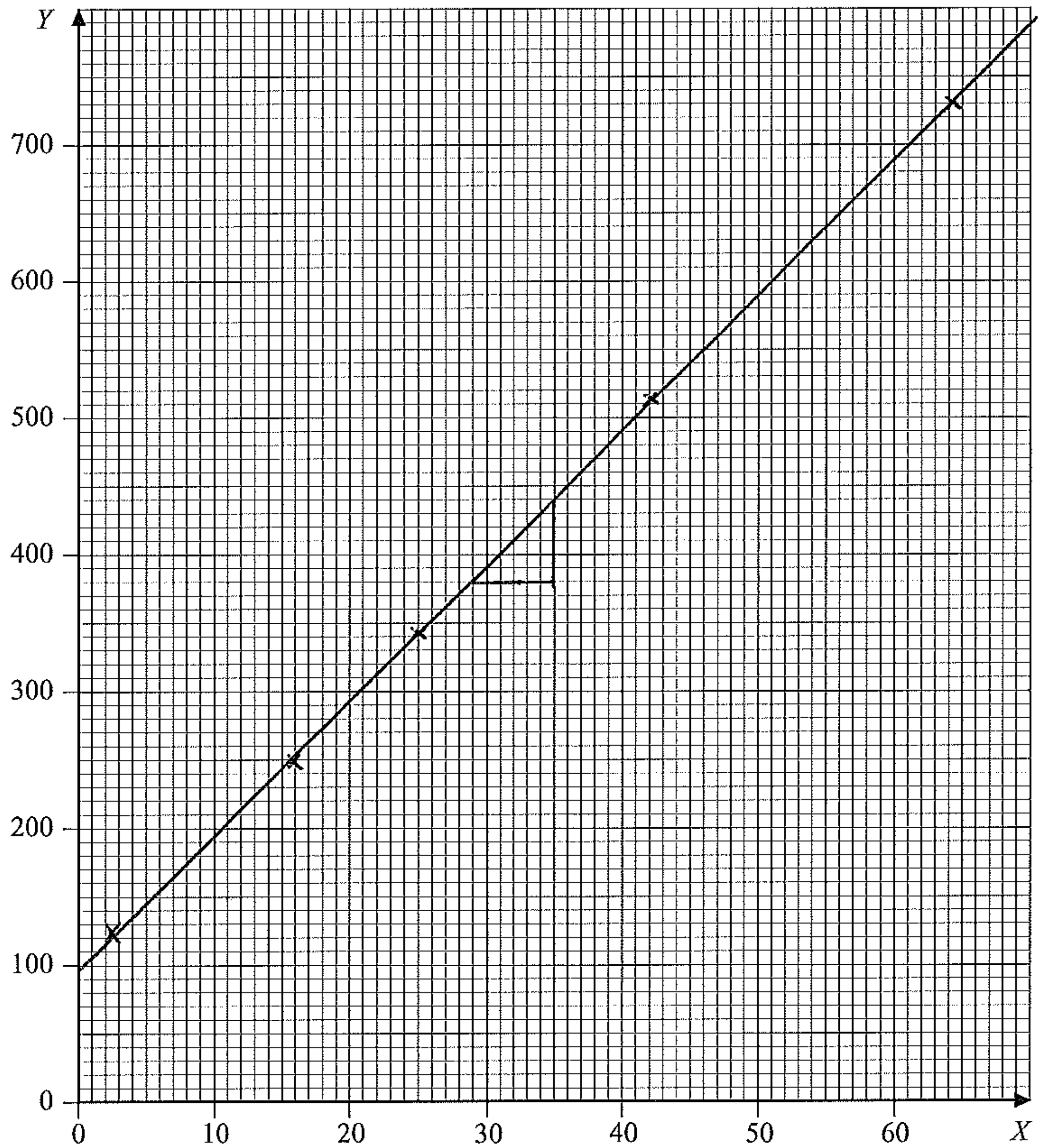


Figure 1 (for Question 7)

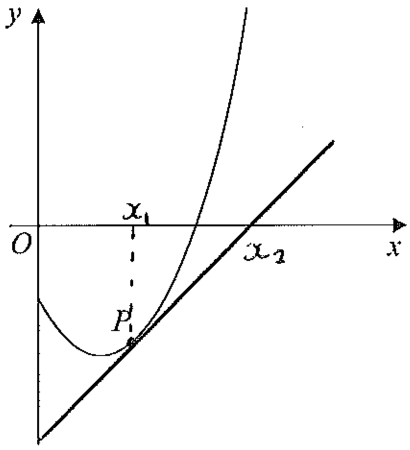


Figure 2 (for Question 8)

(9) a) As $x \to \infty$, $f(x) \to 1/1$ $\therefore y = 1$ is asymptote

when $x = 0$, denominator $= 0$ $\Rightarrow x = 0$ is asymptote

b) i) to intersect x axis, $x^2 + 2x + 2$ needs to $= 0$

Discriminant: $b^2 - 4ac$

$\rightarrow \quad 4 - 4 \times 1 \times 2 = -4$

Negative discriminant means no real solutions, so $\neq 0$

ii) $x = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 2}}{2}$

$= \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

c) i) Let $f(x) = k \quad \rightarrow \quad \dfrac{x^2 + 2x + 2}{x^2} = k$

$x^2 + 2x + 2 = kx^2$

$x^2 - kx^2 + 2x + 2 = 0$

$(1-k)x^2 + 2x + 2 = 0$

For equal roots, $b^2 - 4ac = 0$

$\rightarrow \quad 2^2 - 4(1-k) \times 2 = 0$

$\rightarrow \quad 4 - 8(1-k) = 0$

ii) Solve: $4 - 8(1-k) = 0$

$\rightarrow \quad 4 = 8(1-k) \rightarrow 1/2 = 1 - k \rightarrow k = 1/2$

At stationary points, $y = k = 1/2$

Only 1 solution, so only 1 stationary point

Find x: $\cancel{1/2 = x^2 + 2x + 2}$

($k = 1/2$) $\quad (1-k)x^2 + 2x + 2 = 0$

$1/2\, x^2 + 2x + 2 = 0$

$x^2 + 4x + 4 = 0$

$(x+2)(x+2) = 0$

$\hookrightarrow x = -2 \qquad \therefore$ co-ordinates $(-2, 1/2)$