C4 June 2015

① $\dfrac{19x-2}{(5-x)(1+6x)} = \dfrac{A}{(5-x)} + \dfrac{B}{(1+6x)} \Rightarrow 19x-2 = A(1+6x) + B(5-x)$

let $x=5$ $\Rightarrow$ $93 = A(31) + 0$

so $31A = 93$ $\therefore$ $A=3$

Can now use any value for x.

use $x=0$ $\quad -2 = A(1) + B(5)$

$-2 = 3\times 1 + 5B$

$\Rightarrow$ $5B = -5$

$B = -1$

$\dfrac{3}{5-x} - \dfrac{1}{1+6x}$

(b) $\displaystyle\int \frac{3}{5-x} - \frac{1}{1+6x}\,dx$ $\quad$ use $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)$.

$\displaystyle\int \frac{3}{5-x} = -3\int \frac{-1}{5-x}\,dx = -3\ln|5-x|$

$(+c)$

$\displaystyle -\int \frac{1}{1+6x} = -\frac{1}{6}\int \frac{6}{1+6x}\,dx = -\frac{1}{6}\ln|1+6x|$

$\left[-3\ln(5-x) - \frac{1}{6}\ln(1+6x)\right]_0^4 = \left(\cancel{-3\ln 1} - \frac{1}{6}\ln 25\right) - \left(-3\ln 5 - \cancel{\frac{1}{6}\ln 1}\right)$

$\boxed{\ln 1 = 0}$

$= -\frac{1}{6}\ln 25 + 3\ln 5$

$= -\frac{1}{6}\ln 5^2 + 3\ln 5$

$= -\frac{2}{6}\ln 5 + 3\ln 5$

$= 3\ln 5 - \frac{1}{3}\ln 5$

$= \left(3 - \frac{1}{3}\right)\ln 5$

$= \frac{8}{3}\ln 5$

2) $2\cos x - 5\sin x = R\cos(x+\alpha)$ · use addition formulae.

$= R\cos x\cos\alpha - R\sin x\sin\alpha$.

compare coefficients of $\cos x$ and $\sin x$:

(1) $2 = R\cos\alpha$

(2) $-5 = -R\sin\alpha \Rightarrow 5 = R\sin\alpha$

$R^2 = 2^2+5^2 = 29 \Rightarrow R=\sqrt{29}$

$\frac{(2)}{(1)}$ $\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \frac{5}{2} \Rightarrow \tan\alpha = \frac{5}{2}$

$\alpha = \tan^{-1}\left(\frac{5}{2}\right) = 1.19028995$

so $\alpha = 1.19$ (3SF)

$\therefore \sqrt{29}\cos(x+1.19)$.

(b) max value is when $\cos(x+1.19)=1$

(so max value $=\sqrt{29}$)

$\cos(x+1.19) = 1$

$x+1.19 = \cos^{-1}(1) = 0,\ 2\pi$

$x+1.19 = 0,\ 2\pi$

$x = \cancel{-1.19},\ \underline{5.09}$

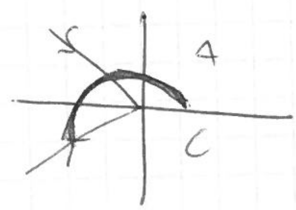
↓ out of range

(ii) $\sqrt{29}\cos(x+1.19)\ \underline{+1} = 0 \Rightarrow \cos(x+1.19) = \frac{-1}{\sqrt{29}}$.

$x+1.19 = \cos^{-1}\left(\frac{-1}{\sqrt{29}}\right) = 1.7575\ldots$

$-1.7575\ldots$

$(2\pi - 1.7575\ldots) \rightarrow 4.525\ldots$



$x = 0.568,\ 3.34 \rightarrow$ (needs to be exactly 3.34 no rounding error allowe

↓

(0.567 is also accepted).

③ $f\left(\frac{-1}{2}\right) = -2$ (remainder theorem!)

$-2 = 8\left(\frac{-1}{2}\right)^3 - 12\left(\frac{-1}{2}\right)^2 - 2\left(\frac{-1}{2}\right) + d$

$-2 = -1 - 3 + 1 + d$

$-2 = -3 + d \quad \Rightarrow \quad \underline{\underline{d = 1}}$

b(i) $g\left(\frac{-1}{2}\right) = 0 \Rightarrow (2x+1)$ is a factor of $g(x)$.

$$\begin{array}{r} 4x^2 - 8x + 3 \\ 2x+1 \overline{)\, 8x^3 - 12x^2 - 2x + 3} \\ -\ 8x^3 + 4x^2 \qquad\qquad \\ \hline -16x^2 - 2x + 3 \\ -\ -16x^2 - 8x \qquad \\ \hline 6x + 3 \\ -\ 6x + 3 \\ \hline 0 \\ \hline \end{array}$$

Factorise quadratic:

$3 \times 4 = 12$

$\overbrace{4x^2 - 8x + 3}$

$\frac{\times}{12} \quad \frac{+}{-8}$

$\underline{\underline{-2, -6}}$

$(4x - 6)(4x - 2)$

$\downarrow \div 2 \qquad \downarrow \div 2$

$(2x - 3)(2x - 1)$

$\therefore (2x+1)(2x-3)(2x-1)$

(ii) $(4x^2 - 1) = (2x+1)(2x-1)$

so $\frac{4x^2 - 1}{g(x)} = \frac{\cancel{(2x+1)}\cancel{(2x-1)}}{\cancel{(2x+1)}(2x-3)\cancel{(2x-1)}} = \frac{1}{(2x-3)} = (2x-3)^{-1}$

$h'(x) = -1 \times 2(2x-3)^{-2} = -2(2x-3)^{-2} = \frac{-2}{(2x-3)^2}$

$\left(\frac{dy}{dx}\right)$

since $(2x-3)^2 > 0$ for all x values

$\Rightarrow \frac{-2}{(2x-3)^2} < 0$ " " " "

$\therefore \frac{dy}{dx} < 0$ for any x value

$\Rightarrow$ decreasing function.

④ $(1+5x)^{\frac{1}{5}}$ $\quad n=\frac{1}{5}$ $\quad$ "x" $= 5x$

(a) $1 + nx + \frac{n(n-1)x^2}{2!} = 1 + \left(\frac{1}{5}\right)(5x) + \frac{\left(\frac{1}{5}\right)\left(\frac{-4}{5}\right)(5x)^2}{2}$

$= 1 + x - 2x^2$

(b)(i) $(8+3x)^{-\frac{2}{3}} = \left[8\left(1+\frac{3}{8}x\right)\right]^{-\frac{2}{3}} = 8^{-\frac{2}{3}}\left(1+\frac{3}{8}x\right)^{-\frac{2}{3}}$

$= \frac{1}{4}\left(1+\frac{3}{8}\right)^{-\frac{2}{3}}$.

$n = \frac{-2}{3}$ $\quad$ "x" $= \frac{3}{8}x$

$\Rightarrow \frac{1}{4}\left[1 + \left(\frac{-2}{3}\right)\left(\frac{3}{8}x\right) + \frac{\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{3}{8}x\right)^2}{2}\right]$

$= \frac{1}{4}\left[1 - \frac{1}{4}x + \frac{5}{64}x^2\right]$

$= \frac{1}{4} - \frac{1}{16}x + \frac{5}{256}x^2$.

(ii) $\sqrt[3]{\frac{1}{81}} = 81^{-\frac{1}{3}} = (9^2)^{-\frac{1}{3}} = 9^{-\frac{2}{3}}$

so let $8 + 3x = 9$.

$\underline{\underline{x = \frac{1}{3}}}$

$\frac{1}{4} + \frac{1}{16}\left(\frac{1}{3}\right) + \frac{5}{256}\left(\frac{1}{3}\right)^2 = 0.231336805̇$

$\Rightarrow \sqrt[3]{\frac{1}{81}} \approx 0.2313$.

⑤ (a) $x = \cos 2t$ $\qquad y = \sin t$

$\frac{dx}{dt} = -2\sin 2t$ $\qquad \frac{dy}{dt} = \cos t$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-2\sin 2t}$ $\qquad$ when $t = \frac{\pi}{6}$

$m = \frac{\cos\left(\frac{\pi}{6}\right)}{-2\sin\left(\frac{2\pi}{6}\right)} = \frac{-1}{2}$

(b) At P: $\quad t = \frac{\pi}{6}$ $\quad$ so $x = \cos\left(\frac{2\pi}{6}\right) = \frac{1}{2}$

$y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

Normal so $m = 2$ $\qquad$ (negative reciprocal of (a)).

$y - \frac{1}{2} = 2\left(x - \frac{1}{2}\right) \Rightarrow y - \frac{1}{2} = 2x - 1$

$y = 2x - \frac{1}{2}$.

(c) At Q: $\quad x = \cos 2q \quad y = \sin q$. $\quad$ Q lies on the normal:

$y = 2x - \frac{1}{2}$

$\sin q = 2\cos 2q - \frac{1}{2}$ $\qquad$ use $\boxed{\cos 2q = 1 - 2\sin^2 q}$

$\sin q = 2(1 - 2\sin^2 q) - \frac{1}{2}$

$\sin q = 2 - 4\sin^2 q - \frac{1}{2}$ $\qquad \Rightarrow \sin q = \frac{3}{2} - 4\sin^2 q$

$\Rightarrow 4\sin^2 q + \sin q - \frac{3}{2} = 0$

$\Rightarrow 8\sin^2 q + 2\sin q - 3 = 0$.

Factorise: $8s^2 + 2s - 3 = 0$ $\qquad$ $\times$: -24 $\quad$ $\pm$: $+2$ $\qquad (+6, -4)$

$(8s + 6)(8s - 4)$

$\div 2 \qquad \div 4$

$(4s + 3) \quad (2s - 1)$

so $(4\sin q + 3)(2\sin q - 1) = 0$. $\qquad$ so $\sin q = \frac{-3}{4}$ or $\sin q = \frac{1}{2}$

this is point P!

so at Q: $\sin q = \frac{-3}{4}$ $\quad$ so $x = \cos 2q = 1 - 2\sin^2 q$

$= 1 - 2\left(\frac{-3}{4}\right)^2 = \frac{-1}{8}$

⑥ (a) $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix}$

Use formula: $\cos\theta = \dfrac{a \cdot b}{|a||b|}$. to find angle between $\begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \downarrow$ gradient vector of l.

$$\begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 6 - 4 + 12 = 14$$

$$\left|\begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix}\right| = \sqrt{2^2+4^2+6^2} = 2\sqrt{14}$$

$$\left|\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}\right| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{14}{2\sqrt{14}\sqrt{14}}\right) = \cos^{-1}\left(\tfrac{1}{2}\right) = \underline{\underline{60^\circ}}$$

(b) C lies on l. So for some value of $\lambda = c$

$$\vec{OC} = \begin{pmatrix} 3+3c \\ 2+c \\ 10-2c \end{pmatrix}$$

$$A\hat{B}C = 90^\circ \Rightarrow \vec{AB} \cdot \vec{BC} = 0$$

$$\vec{BC} = \begin{pmatrix} 3+3c \\ 2+c \\ 10-2c \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2+3c \\ 4+c \\ 6-2c \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -2+3c \\ 4+c \\ 6-2c \end{pmatrix} = 0$$

$$\Rightarrow \begin{array}{r} -4 + 6c \\ -16 - 4c \\ -36 + 12c \\ \hline -56 + 14c \end{array} = 0$$

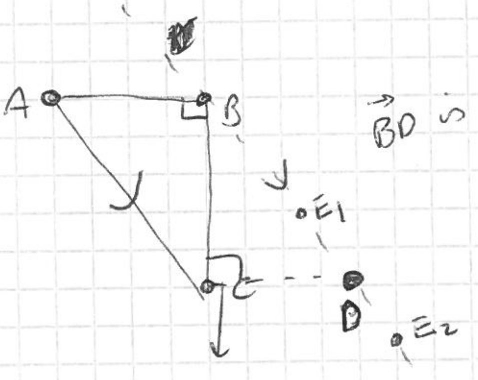
$$-56 + 14c = 0 \Rightarrow 14c = 56$$

$$\underline{\underline{c = 4}}$$

$$\Rightarrow \vec{OC} = \begin{pmatrix} 3+12 \\ 2+4 \\ 10-8 \end{pmatrix}$$

$C(15, 6, 2)$

(c) Need diagram:



$\vec{BD}$ is parallel to $\vec{AC}$

So this is a parallelogram.

$B\hat{C}D = 90°$

O (origin)

$$\vec{OE} = \vec{OD} \pm \frac{1}{2}\vec{AC}$$

$$\vec{OE_1} = \begin{pmatrix}17\\2\\-4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}12\\4\\-8\end{pmatrix} = \begin{pmatrix}17\\2\\-4\end{pmatrix} + \begin{pmatrix}6\\2\\-4\end{pmatrix} = \begin{pmatrix}23\\4\\-8\end{pmatrix}$$

$$\vec{OE_2} = \begin{pmatrix}17\\2\\-4\end{pmatrix} - \frac{1}{2}\begin{pmatrix}12\\4\\-8\end{pmatrix} = \begin{pmatrix}17\\2\\-4\end{pmatrix} - \begin{pmatrix}6\\2\\-4\end{pmatrix} = \begin{pmatrix}11\\0\\0\end{pmatrix}$$

so $E(23, 4, -8)$ or $E(11, 0, 0)$.

(7) when $x = \ln 2$ $\quad y = \frac{1}{2}$

Subs into $y^3 + 2e^{-3x}y - x = k$

$\boxed{b\ln A = \ln A^b}$

$$\Rightarrow \quad k = \left(\tfrac{1}{2}\right)^3 + \cancel{2}e^{-3(\ln 2)}\cancel{\left(\tfrac{1}{2}\right)} - \ln 2$$

$$= \frac{1}{8} + e^{\ln 2^{-3}} - \ln 2$$

$\boxed{e^{\ln A} = A}$

$$= \frac{1}{8} + e^{\ln \frac{1}{8}} - \ln 2$$

$$= \frac{1}{8} + \frac{1}{8} - \ln 2 = \frac{1}{4} - \ln 2.$$

(b) Implicit differentiation

Start with $2e^{-3x}y$ → need product rule.

let $u = 2e^{-3x}$ $\quad v = y$

$\frac{du}{dx} = -6e^{-3x}$ $\quad \frac{dv}{dx} = \frac{dy}{dx}$

$$\Rightarrow \quad 2e^{-3x}\frac{dy}{dx} - 6ye^{-3x}$$

So $y^3 + 2e^{-3x}y - x = k$

$\boxed{\frac{d}{dx}}$

$$3y^2\frac{dy}{dx} + 2e^{-3x}\frac{dy}{dx} - 6ye^{-3x} - 1 = 0$$

$$\frac{dy}{dx}\left(3y^2 + 2e^{-3x}\right) = 1 + 6ye^{-3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 + 6ye^{-3x}}{3y^2 + 2e^{-3x}}$$

At $\underline{P}$ $\quad x = \ln 2$ $\quad y = \frac{1}{2}$ $\quad$ so $\quad \frac{dy}{dx} = \frac{1 + 6\left(\frac{1}{2}\right)e^{-3\ln 2}}{3\left(\frac{1}{2}\right)^2 + 2e^{-3\ln 2}}$

$$= \frac{1 + 3e^{\ln \frac{1}{8}}}{\frac{3}{4} + 2e^{\ln \frac{1}{8}}} = \frac{1 + 3\left(\frac{1}{8}\right)}{\frac{3}{4} + 2\left(\frac{1}{8}\right)} = \frac{\frac{11}{8}}{1} = \frac{11}{8}$$ //

(8) (a) $\dfrac{dx}{dt} = \dfrac{\sqrt{4+5x}}{5(1+t)^2}$ "initially empty" $\Rightarrow t=0 \ \underline{\underline{x=0}}$

Separate variables:

$$\frac{1}{\sqrt{4+5x}} \frac{dx}{dt} = \frac{1}{5(1+t)^2}$$

$$\Rightarrow \int (4+5x)^{-\frac{1}{2}} \, dx = \int \frac{1}{5}(1+t)^{-2} \, dt$$

$$\frac{(4+5x)^{\frac{1}{2}}}{\frac{1}{2} \times 5} = \frac{\frac{1}{5}(1+t)^{-1}}{-1} + c$$

$$\Rightarrow \frac{2}{5}(4+5x)^{\frac{1}{2}} = -\frac{1}{5}(1+t)^{-1} + c$$

$t=0 \quad x=0 \quad \Rightarrow$ use to find c.

$$\frac{2}{5}(4)^{\frac{1}{2}} = -\frac{1}{5}(1)^{-1} + c$$

$$\frac{4}{5} = \frac{-1}{5} + c \qquad \Rightarrow \qquad \underline{\underline{c = 1}}$$

So: $\dfrac{2}{5}(4+5x)^{\frac{1}{2}} = -\dfrac{1}{5}(1+t)^{-1} + 1$

$$(4+5x)^{\frac{1}{2}} = -\frac{1}{2}(1+t)^{-1} + \frac{5}{2} \qquad \left(\times \frac{5}{2}\right)$$

$$4+5x = \left[-\frac{1}{2}(1+t)^{-1} + \frac{5}{2}\right]^2 \qquad \text{(Square both sides)}$$

$$5x = \left[-\frac{1}{2}(1+t)^{-1} + \frac{5}{2}\right]^2 - 4 \qquad (-4)$$

$$x = \frac{1}{5}\left[-\frac{1}{2}(1+t)^{-1} + \frac{5}{2}\right]^2 - \frac{4}{5}. \qquad (\div 5)$$

(b) "rate of change of radius" $\Rightarrow \dfrac{dr}{dt}$

(i) "inversely proportional" $\Rightarrow \dfrac{K}{?}$

"area of surface" $\Rightarrow$ circle radius r $\Rightarrow \pi r^2$

$$\frac{dr}{dt} = \frac{K}{\pi r^2}$$

(or since π is a constant can have $\dfrac{dr}{dt} = \dfrac{K}{r^2}$.)

(ii) when $r=1$ $\dfrac{dr}{dt} = 4 \cdot 5$

so $4 \cdot 5 = \dfrac{K}{\pi \times 1^2} = \dfrac{K}{\pi}$ so $K = 4 \cdot 5 \pi$

$$\frac{dr}{dt} = \frac{4 \cdot 5 \cancel{\pi}}{\cancel{\pi} r^2} = \frac{4 \cdot 5}{r^2}$$

set $\dfrac{dr}{dt} = 0 \cdot 5$

so $0 \cdot 5 = \dfrac{4 \cdot 5}{r^2} \Rightarrow r^2 = \dfrac{4 \cdot 5}{0 \cdot 5} = 9$

$r^2 = 9$

$r = \pm 3$

$r > 0 \Rightarrow \underline{\underline{r = +3}}$