June 2014 C4 Written Solutions

① (a) $x = \frac{1}{2}t^2 + 1$ $\qquad$ $y = 4t^{-1} - 1$

$\frac{dx}{dt} = t$ $\qquad$ $\frac{dy}{dt} = -4t^{-2} = \frac{-4}{t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-4}{t^2}}{t} = \frac{-4}{t^3}$$

$t = 2 \Rightarrow \frac{dy}{dx} = \frac{-4}{2^3} = -\frac{1}{2}$

(b) $y = \frac{4}{t} - 1 \Rightarrow y + 1 = \frac{4}{t} \Rightarrow t = \frac{4}{y+1}$

sub into x:

$$x = \frac{t^2}{2} + 1 = \frac{1}{2}\left(\frac{4}{y+1}\right)^2 + 1$$

$x = \frac{1}{2}\left(\frac{4}{y+1}\right)^2 + 1$ or any equivalent form

eg. $\frac{8}{x-1} = (y+1)^2$ or $8 = (x-1)(y+1)^2$

② (a) $\frac{4x^3 - 2x^2 + 16x - 3}{2x^2 - x + 2} = Ax + \frac{B(4x-1)}{2x^2 - x + 2}$

$\Rightarrow 4x^3 - 2x^2 + 16x - 3 = Ax(2x^2 - x + 2) + B(4x - 1)$

let $x = 0 \Rightarrow -3 = 0 + B(-1) \Rightarrow \boxed{B = 3}$

$-3 = -B$

could use $x = \frac{1}{4}$ ~~or~~ but easier to use

$x = 1$ $\quad 4(1) - 2(1) + 16(1) - 3 = A(2 - 1 + 2) + \overset{(=3)}{B}(4 - 1)$

$15 = 3A + 9 \Rightarrow 6 = 3A \Rightarrow \boxed{A = 2}$

$$\Rightarrow 2x + \frac{3(4x-1)}{2x^2 - x + 2}$$

(b) $y = \int \frac{4x^3 - 2x^2 + 16x - 3}{2x^2 - x + 2}\,dx = \int 2x + 3\int \frac{4x-1}{2x^2-x+2}\,dx$

notice that

$\frac{d}{dx}(2x^2 - x + 2) = 4x - 1$

so $y = \frac{2x^2}{2} + 3\ln|2x^2 - x + 2| + c$

$= x^2 + 3\ln|2x^2 - x + 2| + c$

use $(-1, 2)$

$2 = (-1)^2 + 3\ln|2(-1)^2 - (-1) + 2| + c$

$2 = 1 + 3\ln 5 + c \Rightarrow \underline{c = 1 - 3\ln 5}$

so $y = x^2 + 3\ln|2x^2 - x + 2| + 1 - 3\ln 5$

(3) $1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$ $\quad (n = \frac{1}{4}, \text{"}x\text{"} = -4x)$

$(1-4x)^{\frac{1}{4}} \approx 1 + \left(\frac{1}{4}\right)(-4x) + \frac{\left(\frac{1}{4}\right)\left(1 - \frac{1}{4}\right)(-4x)^2}{2!}$

$= 1 - x + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)16x^2}{2} = \underline{1 - x - \frac{3}{2}x^2}$

(b) $(2+3x)^{-3} = \left[2\left(1 + \frac{3}{2}x\right)\right]^{-3} = 2^{-3}\left(1 + \frac{3}{2}x\right)^{-3} = \frac{1}{8}\left(1 + \frac{3}{2}x\right)^{-3}$

$\begin{pmatrix} n = -3 \\ \text{"}x\text{"} = \frac{3}{2}x \end{pmatrix}$ $\quad \approx \frac{1}{8}\left[1 + (-3)\left(\frac{3}{2}x\right) + \frac{(-3)(-4)\left(\frac{3}{2}x\right)^2}{2}\right]$

$= \frac{1}{8}\left[1 - \frac{9}{2}x + \frac{27}{2}x^2\right] = \underline{\frac{1}{8} - \frac{9}{16}x + \frac{27}{16}x^2}$

(c) $\frac{(1-4x)^{\frac{1}{4}}}{(2+3x)^3} = (1-4x)^{\frac{1}{4}}(2+3x)^{-3}$

$\approx \left(1 - x - \frac{3}{2}x^2\right)\left(\frac{1}{8} - \frac{9}{16}x + \frac{27}{16}x^2\right)$ up to $\underline{x^2 \text{ only}}$

$= \frac{1}{8} - \frac{9}{16}x + \frac{27}{16}x^2$
$\quad - \frac{1}{8}x + \frac{9}{16}x^2 - \frac{3}{16}x^2$ $\quad = \boxed{\frac{1}{8} - \frac{11}{16}x + \frac{33}{16}x^2}$

④ 1 April 2001 $\quad t=0 \quad V=5000$

(a) A is initial value $\quad 5000 = Ap^0 = A$

$A = 5000$

b) (i) 2001 → 2011 $\quad t=10 \quad V=25000$

$25000 = 5000p^{10} \Rightarrow \frac{25000}{5000} = p^{10} \Rightarrow \frac{25}{5} = p^{10}$

So $p^{10} = 5$ (as required)

(ii) $V = 75000$

$75000 = 5000p^t \Rightarrow 15 = p^t \Rightarrow \ln 15 = \ln p^t$

$\ln 15 = t\ln p$ so $t = \frac{\ln 15}{\ln p}$

Use $p^{10} = 5$

$\Rightarrow \ln p^{10} = \ln 5 \Rightarrow \boxed{10\ln p = \ln 5}$ remember this!

So $\ln p = \frac{1}{10}\ln 5$

$\Rightarrow t = \frac{\ln 15}{\frac{1}{10}\ln 5} = 10\frac{\ln 15}{\ln 5} = 16.826$ years.

16.826 years = 16 years and (0.826×12) months

16 years and 9.9 months

round to 10.

1 April 2001 + 16 years & 10 months ⇒ Feb 2018

$\Rightarrow \boxed{2018}$

(c) $W = 2500q^t$ 1 April 1991 $\rightarrow t = 0$

$\underline{V = W}$ $\rightarrow$ be careful with "t".

t_1 = time after 2001

$\underline{t_2 = t}$ = time after 1991

$t_1 = t - 10$ (less time has passed since 2001)

so $\boxed{\begin{aligned} V &= 5000p^{t-10} \\ W &= 2500q^t \end{aligned}}$

$V = W$ at $\underline{t = T}$ $\Rightarrow$ $5000p^{t-10} = 2500q^t$

$2p^{T-10} = q^T$

$\ln\left[2p^{(T-10)}\right] = \ln q^T$

$\Rightarrow \ln 2 + \ln p^{T-10} = T\ln q$

$\ln 2 + (T-10)\ln p = T\ln q$

$\ln 2 + T\ln p \boxed{-10\ln p} = T\ln q$

in b(ii) $\rightarrow$ showed $\boxed{\underline{10\ln p = \ln 5}}$

so $\ln 2 + T\ln p - \ln 5 = T\ln q$

$T\ln p - T\ln q = \ln 5 - \ln 2$ $\Rightarrow T(\ln p - \ln q) = \ln\left(\frac{5}{2}\right)$

$T\ln\left(\frac{p}{q}\right) = \ln\left(\frac{5}{2}\right)$

$\Rightarrow T = \dfrac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{p}{q}\right)}$

as required

(iii) $p = 1.029q$

$$T = \frac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{1.029\cancel{q}}{\cancel{q}}\right)} = \frac{\ln 2.5}{\ln(1.029)} = 32.05 \text{ years.}$$

(0.05 years is less than 1 month)

So 32 years after 1991 (1 April)

$\Rightarrow$ 2023

⑤(a)(i) let $3\sin x \pm 4\cos x = R\sin(x+\alpha)$

use $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$3\sin x \pm 4\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$.

coefficients of $\sin x$ and of $\cos x$:

① $3 = R\cos\alpha$

② $4 = R\sin\alpha$.

Do $\frac{②}{①} \Rightarrow \frac{4}{3} = \frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha}$

$\Rightarrow \frac{4}{3} = \tan\alpha$.

$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$ (→ store full answer in your calculator for use later in question).

$R = \sqrt{3^2+4^2} = 5$

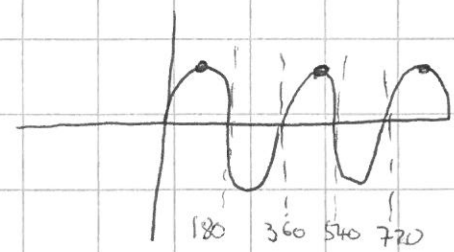
So $5\sin(x+53.1^\circ)$

(ii) let $x = 2\theta$ $\qquad 5\sin(2\theta + 53\cdot1) = 5$

$0 < \theta < 360$

$0 < 2\theta < 720$

$53\cdot1 < (2\theta + 53\cdot1) < 773\cdot1^\circ$



$\sin(2\theta + 53\cdot1) = 1$

$2\theta + 53\cdot1 = \sin^{-1}(1) = 90^\circ, 450^\circ, \cancel{810^\circ}$

$2\theta = 36\cdot87, \quad 396\cdot87$

$\theta = 18\cdot435, \quad 198\cdot435$

$\theta = 18\cdot4^\circ, \quad 198\cdot4^\circ$

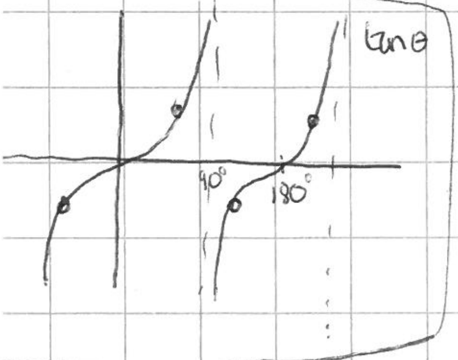
(MS accepts $18\cdot5^\circ$, $198\cdot5^\circ$)

5 b(i) $2\tan^2\theta = 1 \Rightarrow \tan^2\theta = \frac{1}{2}$ so $\tan\theta = \pm\sqrt{\frac{1}{2}}$

$= \pm\frac{\sqrt{2}}{2}$

$\theta = \tan^{-1}\left(\frac{+\sqrt{2}}{2}\right) = 35\cdot26, \quad \tan^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -35\cdot26$

$35\cdot3^\circ \qquad -35\cdot3^\circ$



$180 \pm 35\cdot26$

$35\cdot3^\circ, \ 144\cdot7^\circ, \ \cancel{215\cdot3^\circ}$

c(i) let $f(x) = 8x^3 - 4x + 1$

$f(\tfrac{1}{2}) = 8(\tfrac{1}{2})^3 - 4(\tfrac{1}{2}) + 1$

$= 8(\tfrac{1}{8}) - \frac{4}{2} + 1 = 1 - 2 + 1 = 0$

$f(\tfrac{1}{2}) = 0 \quad \therefore (2x-1)$ is a factor of $f(x)$.

5c(ii) $4\cos 2\theta \cos\theta + 1$

use $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

$\Rightarrow 4\cos 2\theta\cos\theta + 1 = 4(2\cos^2\theta - 1)\cos\theta + 1$

$= 8\cos^3\theta - 4\cos\theta + 1$

Let $x = \cos\theta \Rightarrow \underline{\underline{8x^3 - 4x + 1}}$

(iii) $8x^3 - 4x + 1 = 0$

from (i) we know $(2x-1)$ is a factor.

$$
\begin{array}{r}
4x^2 + 2x - 1 \\
2x-1\,\overline{)\,8x^3 + 0x^2 - 4x + 1} \\
-\ 8x^3 - 4x^2 \quad\quad\quad \\
\hline
-\ 4x^2 - 4x + 1 \\
4x^2 - 2x \quad \\
\hline
-2x + 1 \\
-\ -2x + 1 \\
\hline
0
\end{array}
$$

$(2x-1)(4x^2+2x-1) = 0$

$\Rightarrow 2x - 1 = 0$

$x = \frac{1}{2}$

$\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 60°$

$\downarrow$

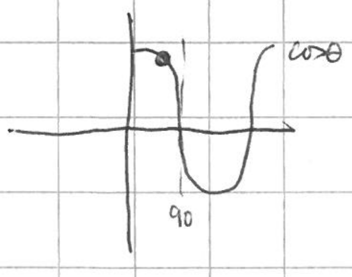
we are looking for cos 72.

$x = \cos 72$ must be a solution to

$4x^2 + 2x - 1 = 0$

use equation $a = 4$ $b = 2$ $c = -1$

$$x = \frac{-2\sqrt{(2)^2 - 4(4)(-1)}}{2(4)} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$



$\underline{\underline{\cos 72 > 0}}$ $\Rightarrow x = \cos 72 = \dfrac{-1+\sqrt{5}}{4}$

5 (a) $\vec{PQ} = \vec{OQ} - \vec{OP}$

$\vec{OQ}$: Sub $\mu = 2$ into l_2

$$l_2 : \quad r = \begin{pmatrix} 7+2\mu \\ -8+3\mu \\ 6+\mu \end{pmatrix} \Rightarrow \vec{OQ} = \begin{pmatrix} 7+2(2) \\ -8-3(2) \\ 6+(2) \end{pmatrix} = \begin{pmatrix} 11 \\ -14 \\ 8 \end{pmatrix}$$

$\vec{OP}$: sub $\lambda = -1$ into l_1

$$l_1 : \quad r = \begin{pmatrix} 4-\lambda \\ -5+3\lambda \\ 3+\lambda \end{pmatrix} \Rightarrow \vec{OP} = \begin{pmatrix} 4-(-1) \\ -5+3(-1) \\ 3+(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ 2 \end{pmatrix}$$

so $$\vec{PQ} = \begin{pmatrix} 11 \\ -14 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ -8 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix}$$

$$= 6\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$ $\therefore$ $\vec{PQ}$ is parallel to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

b) There is a value of λ and μ where

$$\begin{pmatrix} 7+2\mu \\ -8-3\mu \\ 6+\mu \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ -5+3\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} 3 \\ b \\ c \end{pmatrix}$$

i

$7+2\mu = 3$ and $4-\lambda = 3$

$2\mu = -4$ $\qquad$ $\lambda = 1$

$\mu = -2$

j

sub in $\mu = -2$ $\qquad$ check $\lambda = 1$

$-8-3(-2) = -2$ $\qquad$ $-5+3\lambda = -2$ ✓

$b = -2$

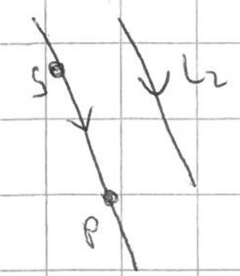
k

$6+(-2) = 4$ $\qquad$ check $3+(+1) = 4$ ✓

$c = 4$

$R\ (3, -2, 4)$

b(ii)



Find the equation of the line through $P(5,-8,2)$ which is parallel to L_2

$$L_3: \quad r = \begin{pmatrix} 5 \\ -8 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 5 \\ -8 \\ 2 \end{pmatrix} \to \vec{OP}$

$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \to$ gradient / directional vector of line L_2.

So $\vec{OS} = \begin{pmatrix} 5+2t \\ -8-3t \\ 2+t \end{pmatrix}$ for some value of t since S lies on this line.

$\vec{RS} \cdot \vec{PQ} = 0$ since $\vec{RS}$ $\underline{\perp}$ to $\vec{PQ}$

$$\vec{RS} = \vec{OS} - \vec{OR} = \begin{pmatrix} 5+2t \\ -8-3t \\ 2+t \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+2t \\ -6-3t \\ -2+t \end{pmatrix}$$

So $\vec{PQ} \cdot \vec{RS} = 0$

$$\Rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2+2t \\ -6-3t \\ -2+t \end{pmatrix} = 0 \quad \Rightarrow \begin{matrix} 2+2t \\ 6+3t \\ \underline{-2+t} \end{matrix} = 0$$

$$6 + 6t = 0$$

$$\Rightarrow 6t = -6$$

and $\underline{\underline{t = -1}}$

$$\text{So} \quad \vec{OS} = \begin{pmatrix} 5+2(-1) \\ -8-3(-1) \\ 2+(-1) \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$$

So $\underline{\underline{S(3,-5,1)}}$.

7) a(i) $\cos 2y + ye^{3x} = 2\pi$

$$\frac{d}{dx}(\cos 2y) + \frac{d}{dx}(ye^{3x}) = \underbrace{\frac{d}{dx}(2\pi)}_{=0}$$

$$\frac{d}{dx}(\cos 2y) = \frac{dy}{dx} \times \frac{d}{dy}(\cos 2y) = -2\sin 2y$$

$$\frac{d}{dx}(ye^{3x}) \quad \left(\begin{array}{ll} \text{use product rule} & \\ u = y & v = e^{3x} \\ \frac{du}{dx} = \frac{dy}{dx} & \frac{dv}{dx} = 3e^{3x} \end{array}\right) \quad \underline{u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$= 3ye^{3x} + \frac{dy}{dx}e^{3x}$$

$$\Rightarrow -2\sin 2y\frac{dy}{dx} + 3ye^{3x} + e^{3x}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(e^{3x} - 2\sin 2y) = -3ye^{3x}$$

$$\frac{dy}{dx} = \frac{-3ye^{3x}}{e^{3x} - 2\sin 2y} \quad \left(= \frac{3ye^{3x}}{2\sin 2y - e^{3x}}\right)$$

(ii) $x = \ln 2$, $y = \frac{\pi}{4}$

$$\Rightarrow \frac{dy}{dx} = \frac{3\left(\frac{\pi}{4}\right)e^{3\ln 2}}{2\sin\left(\frac{2\pi}{4}\right) - e^{3\ln 2}}$$

$$e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8, \quad \text{and} \quad \sin\left(\frac{2\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{3\pi}{4} \times 8}{2 - 8} = \frac{6\pi}{-6} = -\pi$$

(b) Normal at A has gradient $\frac{1}{\pi}$

$A\left(\ln 2, \frac{\pi}{4}\right)$

$\Rightarrow \quad y - \frac{\pi}{4} = \frac{1}{\pi}(x - \ln 2)$

y-intercept $\Rightarrow x = 0$ so $y - \frac{\pi}{4} = \frac{1}{\pi}(-\ln 2)$

$$\boxed{y = \frac{\pi}{4} - \frac{\ln 2}{\pi}}$$

(8) $\frac{16}{(1-3x)(1+x)^2} = \frac{A}{1-3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$ $\Big| \left(\times (1-3x)(1+x)^2\right)$

$16x = A(1+x)^2 + B(1-3x)(1+x) + C(1-3x)$

let $x = -1$ $\Rightarrow$ $-16 = 0 + 0 + C(1-3(-1))$

$-16 = 4C \Rightarrow \boxed{C = -4}$

let $x = \frac{1}{3}$ $\Rightarrow$ $\frac{16}{3} = A\left(\frac{4}{3}\right)^2 + 0 + 0$

$\Rightarrow \frac{16}{3} = A \times \frac{16}{9} \Rightarrow A = \frac{9}{3} = 3$

$\boxed{A = 3}$

let $x = 0$ $\quad 0 = A(1)^2 + B(1)(1) + C(1)$

$0 = 3 + B - 4 \Rightarrow 0 = B - 1$

$\Rightarrow \boxed{B = 1}$

$$\boxed{\frac{3}{1-3x} + \frac{1}{1+x} + \frac{-4}{(1+x)^2}}$$

8(b) $\frac{dy}{dx} = e^{2y}\left[\frac{3}{1-3x} + \frac{1}{1+x} + \frac{-4}{(1+x)^2}\right]$

Separate variables.

$$\int \frac{1}{e^{2y}}\,dy = \int \frac{3}{1-3x}\,dx + \int \frac{1}{1+x}\,dx - 4\int \frac{1}{(1+x)^2}\,dx$$

$$\int e^{-2y} = -1\int \frac{-3}{1-3x}\,dx + \int \frac{1}{1+x}\,dx - 4\int (1+x)^{-2}\,dx$$

$$-\frac{1}{2}e^{-2y} = -\ln|1-3x| + \ln|1+x| - 4\left[-(1+x)^{-1}\right] + c$$

$(\times -2)$

$$e^{-2y} = 2\ln|1-3x| - 2\ln|1+x| - \frac{8}{1+x} + d$$

$\underline{x=0}$ $\underline{y=0}$

$$e^{0} = 2\ln 1 - 2\ln 1 - \frac{8}{1} + d$$

$$1 = -8 + d \Rightarrow d = 9$$

$$\therefore \quad e^{-2y} = 2\ln|1-3x| - 2\ln|1+x| - \frac{8}{1+x} + 9$$

If you didn't do this, you get

$$c = -\frac{9}{2}$$

and $-\frac{1}{2}e^{-2y} = -\ln|1-3x| + \ln|1+x| + \frac{4}{1+x} - \frac{9}{2}$