Core 4 Jan 2013 Written Solutions

① (a) $f(x) = 2x^3 + x^2 - 8x - 7$

$f\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right)^3 + \left(\frac{-1}{2}\right)^2 - 8\left(\frac{-1}{2}\right) - 7$

$= \frac{-2}{8} + \frac{1}{4} + 4 - 7 = -3$

(b)(i) $g(x) = f(x) + d = 2x^3 + x^2 - 8x + (d-7)$

$g\left(\frac{-1}{2}\right) = 0$ since $(2x+1)$ is a factor.

$2\left(\frac{-1}{2}\right)^3 + \left(\frac{-1}{2}\right)^2 - 8\left(\frac{-1}{2}\right) + d - 7 = 0$

$\frac{-1}{4} + \frac{1}{4} + 4 + d - 7 = 0$

$-3 + d = 0$ (Using part (a) you can get to this line immediately)

$\Rightarrow \boxed{d = 3}$

$\Rightarrow g(x) = 2x^3 + x^2 - 8x - 4$

(ii) $y = g(x) = (2x+1)(x^2+a)$

Consider constants: $-4 = 1 \times a \Rightarrow a = -4$

$g(x) = (2x+1)(x^2-4) = (2x+1)(x+2)(x-2)$

(iii) $2x^3 - 3x^2 - 2x = x(2x^2 - 3x - 2)$

$= x(2x+1)(x-2)$

$\frac{\cancel{(2x+1)}(x+2)\cancel{(x-2)}}{x\cancel{(2x+1)}\cancel{(x-2)}} = \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + \frac{2}{x}$

2 (a) $\dfrac{7x-1}{(1+3x)(3-x)} = \dfrac{A}{\cancel{1+3x}} + \dfrac{B}{1+3x}$

$\times (1+3x)(3-x)$

$7x-1 = A(1+3x) + B(3-x)$ (*)

$7x-1 = A + 3Ax + 3B - Bx$

$= (3A-B)x + 3B + A$

equate coefficients: $7 = 3A - B$ (1)

$-1 = A + 3B$ (2)

$$\begin{array}{rcll} & 7 & = 3A - B & \\ - & -3 & = 3A + 9B & (2) \times 3 \\ \hline & 10 & = -10B & \end{array}$$

$B = -1 \quad \Rightarrow \quad A = 2$

Alt
quicker method:
sub $x = 3$
then $x = -\frac{1}{3}$
into (*) to
find A and B

$f(x) = \dfrac{2}{3-x} - \dfrac{1}{1+3x}$

(b)(i) $f(x) = 2(3-x)^{-1} - (1+3x)^{-1}$

$2(3-x)^{-1}$

$= 2\left[3\left(1-\frac{x}{3}\right)\right]^{-1} = \frac{2}{3}\left(1-\frac{x}{3}\right)^{-1} \approx \frac{2}{3}\left(1 + (-1)\left(\frac{-x}{3}\right) + \frac{(-1)(-2)}{2!}\left(\frac{-x}{3}\right)^2\right)$

$= \frac{2}{3}\left(1 + \frac{x}{3} + \frac{x^2}{9}\right) = \frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2$

$(1+3x)^{-1} \approx 1 + (-1)(3x) + \frac{(-1)(-2)}{2!}(3x)^2 = 1 - 3x + 9x^2$

$$f(x) \approx \begin{array}{l} \frac{2}{3} + \frac{2}{9}x + \frac{2}{27}x^2 \\ - \quad 1 - 3x + 9x^2 \end{array} = \frac{-1}{3} + \frac{29}{9}x - \frac{241}{27}x^2$$

(ii) only valid when $\left|\frac{x}{3}\right| < 1$ and $|3x| < 1$

$\Rightarrow |x| < \frac{1}{3}$

$x = 0.4$ lies out of this range.

$0.4 > \frac{1}{3}$.

③ (a) (i) $3\cos x + 2\sin x = R\cos(x - \alpha)$

$\cos(x - \alpha) = \cos x \cos\alpha + \sin x \sin\alpha$

$3\cos x + 2\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$.

equate coefficients of $\cos x$ and $\sin x$

① $3 = R\cos\alpha$ ② $2 = R\sin\alpha$.

$\frac{②}{①} \Rightarrow \frac{2}{3} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha \Rightarrow \alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.7^\circ$

$\cancel{R = \frac{3}{\cos\alpha}}$ $①^2 + ②^2 \Rightarrow$

$3^2 + 2^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$

$\Rightarrow 13 = R^2(\cos^2\alpha + \sin^2\alpha) \Rightarrow R = \sqrt{13}$

(ii) $\sqrt{13}\cos(x - 33.7^\circ)$

Min value $= -\sqrt{13}$ occurs when $\cos(x - 33.7^\circ) = -1$

$x - 33.7^\circ = 180^\circ$

$x = 213.7^\circ$

(b)(i) LHS:

$\cot x - \sin 2x$

$= \cot x - 2\cos x \sin x$

(using double-angle)

RHS:

$\cot x \cos 2x = \cot x(\cos^2 x - \sin^2 x)$

(double-angle)

$= \cot x(1 - 2\sin^2 x)$

$= \cot x - 2\cot x \sin^2 x$

$= \cot x - \frac{2\cos x}{\sin x}\sin^2 x$

$= \cot x - 2\cos x \sin x)$

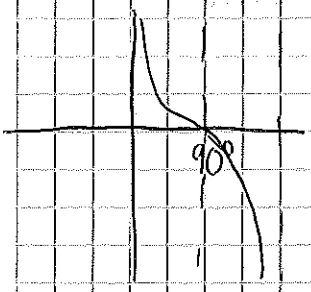
(use $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$)

LHS = RHS ✓

(ii) $\cot x - \sin 2x = \cot x \cos 2x$

$\Rightarrow$ solve $\cot x \cos 2x = 0$

$\cot x = 0$



$\underline{x = 90°}$

$\cos 2x = 0$

$2x = 90°, 270°$

$\underline{x = 45°, 135°}$

4 (a) (i) $x^2 - y^2 = 8$

$$\frac{d}{dx}x^2 - \frac{d}{dx}y^2 = \frac{d}{dx}8$$

$$2x - \frac{d}{dy}y^2\frac{dy}{dx} = 0$$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

so at (p, q) $\frac{dy}{dx} = \frac{p}{q}$.

(ii) $\frac{dy}{dx} = \frac{p}{q}$ | $\frac{dy}{dx} = \frac{-p}{q}$

$(y - q) = \frac{p}{q}(x - p)$ | $(y + q) = \frac{-p}{q}(x - p)$

$$\begin{aligned} & y = q + \frac{p}{q}(x - p) \\ + \quad & y = -q - \frac{p}{q}(x - p) \\ \hline & 2y = 0 \end{aligned}$$

$\Rightarrow y = 0$ (ie x-axis).

(b) $x^2 = \left(t + \frac{2}{t}\right)^2 = t^2 + 4 + \frac{4}{t^2}$

$y^2 = \left(t - \frac{2}{t}\right)^2 = t^2 - 4 + \frac{4}{t^2}$

$x^2 - y^2 = 4 - (-4) = 8$

$x^2 - y^2 = 8$ as required.

(5) (a) $\int x(x^2+3)^{\frac{1}{2}} \, dx$

$$\frac{d}{dx}(x^2+3)^{\frac{3}{2}} = \frac{3}{2}(2x)(x^2+3)^{\frac{1}{2}}$$
$$= 3x(x^2+3)^{\frac{1}{2}}$$

$$\int x(x^2+3)^{\frac{1}{2}} \, dx = \frac{1}{3}\int 3x(x^2+3)^{\frac{1}{2}} \, dx = \frac{1}{3}(x^2+3)^{\frac{3}{2}} + C.$$

(b) $\frac{dy}{dx} = \frac{x\sqrt{x^2+3}}{e^{2y}}$

Separate variables:

$$\int e^{2y} \, dy = \int x\sqrt{x^2+3} \, dx$$

$$\frac{1}{2}e^{2y} = \frac{1}{3}(x^2+3)^{\frac{3}{2}} + C \quad (\text{using (a)})$$

Use fact that $y=0$ when $x=1$

$$\frac{1}{2}(1) = \frac{1}{3}(4)^{\frac{3}{2}} + C$$

$$\frac{1}{2} = \frac{8}{3} + C \quad \Rightarrow \quad C = -\frac{13}{6}$$

$$\frac{1}{2}e^{2y} = \frac{1}{3}(x^2+3)^{\frac{3}{2}} - \frac{13}{6}$$

$$e^{2y} = \frac{2}{3}(x^2+3)^{\frac{3}{2}} - \frac{13}{3}$$

$$2y = \ln\left[\frac{2}{3}(x^2+3)^{\frac{3}{2}} - \frac{13}{3}\right]$$

$$y = \frac{1}{2}\ln\left[\frac{2}{3}(x^2+3)^{\frac{3}{2}} - \frac{13}{3}\right]$$

6 (a)(i) $\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 8 \\ -4 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(ii) let angle ACB $= \theta$

$\vec{AC} \cdot \vec{BC} = |AC||BC| \cos\theta \qquad \Rightarrow \boxed{\cos\theta = \frac{\vec{AC} \cdot \vec{BC}}{|AC||BC|}}$

$\vec{BC} = \begin{pmatrix} 8 \\ -4 \\ -6 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$

$5\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} = 15 + 10 = \underline{25}$

$|AC| = \sqrt{5^2 + (-5)^2} = 5\sqrt{2} \qquad |BC| = \sqrt{3^2 + (-2)^2 + (-6)^2} = \sqrt{49} = 7$

$\Rightarrow \cos\theta = \frac{25}{5\sqrt{2} \times 7} = \frac{5}{7\sqrt{2}} = \frac{5\sqrt{2}}{14}$

so $\theta = \cos^{-1}\left(\frac{5\sqrt{2}}{14}\right)$

(b) $\vec{AC}$: $\quad r = \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(c) $\vec{OD} = \begin{pmatrix}6\\-1\\p\end{pmatrix}$ $\quad \vec{AC}$: $r = \begin{pmatrix}3+\lambda\\1-\lambda\\-6\end{pmatrix}$

(i) BD : $s = \begin{pmatrix}5\\-2\\0\end{pmatrix} + \mu\begin{pmatrix}6-5\\-1+2\\p\end{pmatrix} = \begin{pmatrix}5+\mu\\-2+\mu\\\mu p\end{pmatrix}$

$\vec{BD} = \vec{OD} - \vec{OB}$

We know that r and s intersect so there is a point where:

$3+\lambda = 5+\mu$ (1)

$1-\lambda = -2+\mu$ (2)

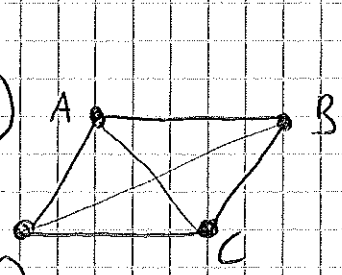
$-6 = \mu p$ (3)

(1) + (2) : $4 = 3 + 2\mu$

$\Rightarrow \boxed{\mu = \frac{1}{2}}$

(3) : $-6 = \frac{1}{2}p \Rightarrow \boxed{p = -12}$ point $D = (6, -1, -12)$

(ii)



We need to show that $|AB| = |BC| = |CD| = |AD|$

$\vec{AB} = \begin{pmatrix}2\\-3\\6\end{pmatrix}$ $\quad \vec{BC} = \begin{pmatrix}3\\-2\\-6\end{pmatrix}$ $\quad \vec{CD} = \begin{pmatrix}-2\\3\\-6\end{pmatrix}$ $\quad \vec{AD} = \begin{pmatrix}3\\-2\\-6\end{pmatrix}$

$|AB|^2 = |BC|^2 = |CD|^2 = |AD|^2 = 2^2 + 3^2 + 6^2 = 49$

$\Rightarrow ABCD$ is a rhombus with sides of length

$\sqrt{49} = 7$

(7)

(a)(i) $\underline{t=0}$ $\quad N = \frac{500}{2} = \underline{\underline{250}}$

(ii) $N = \frac{500}{1+9e^{-3}} = 345.28 \Rightarrow \underline{\underline{345}}$

(iii) $400 = \frac{500}{1+9e^{-\frac{T}{8}}}$

$$400\left(1+9e^{-\frac{T}{8}}\right) = 500$$

$$9e^{-\frac{T}{8}} = \frac{500}{400} - 1 = \frac{1}{4}$$

$$e^{-\frac{T}{8}} = \frac{1}{36}$$

$$-\frac{T}{8} = \ln\left(\frac{1}{36}\right) \Rightarrow T = -8\ln\left(\frac{1}{36}\right)$$

$$= 8\ln\left(\frac{1}{36}\right)^{-1}$$

$$\underline{\underline{T = 8\ln 36}}$$

(b) $\frac{dN}{dt} = -500 \times \left(\frac{-9}{8}e^{-\frac{t}{8}}\right)\left(1+9e^{-\frac{t}{8}}\right)^{-2}$

Look at the form we need it in - there are no exponentials and N appears.

If $N = 500\left(1+9e^{-\frac{t}{8}}\right)^{-1} \Rightarrow \boxed{1+9e^{-\frac{t}{8}} = \frac{500}{N}}$

$$\Rightarrow \boxed{9e^{-\frac{t}{8}} = \frac{500}{N} - 1}$$

$$\frac{dN}{dt} = -500 \times \frac{-1}{8}\left[9e^{-\frac{t}{8}}\right]\left[1+9e^{-\frac{t}{8}}\right]^{-2}$$

$$= -500 \times \frac{-1}{8}\left[\frac{500}{N} - 1\right]\left[\frac{500}{N}\right]^{-2}$$

$\longrightarrow$

$$= \frac{500}{8}\left(\frac{500}{N}-1\right)\left(\frac{N}{500}\right)^2$$

$$= \frac{N^2}{4000}\left(\frac{500}{N}-1\right)$$

$$= \frac{N}{4000}\left(500-N\right)$$

(iii) $\frac{dN}{dt}$ max value.

To find the value of t at which $\frac{dN}{dt}$ is max you would find where $\frac{d^2N}{dt^2}=0$

Easier method:

To find the value of N at which $\frac{dN}{dt}$ is max we can find where $\frac{d}{dN}\left(\frac{dN}{dt}\right)=0$

$$\frac{d}{dN}\left(\frac{N}{4000}(500-N)\right) = \frac{d}{dN}\left(\frac{500}{4000}N-\frac{N^2}{4000}\right)$$

$$= \frac{500}{4000}-\frac{2}{4000}N$$

$\frac{1}{4000}(500-2N)=0$ when $500=2N$

$$\underline{\underline{N=250}}$$

$$250 = \frac{500}{1+9e^{-\frac{t}{8}}} \Rightarrow 2 = 1+9e^{-\frac{t}{8}}$$

$$\frac{1}{9} = e^{-\frac{t}{8}}$$

So $\ln\frac{1}{9} = -\frac{t}{8} \Rightarrow t = -8\ln\frac{1}{9} = 8\ln\left(\frac{1}{9}\right)^{-1} = \underline{\underline{8\ln 9}}$

Centre Number						Candidate Number				
Surname										
Other Names										
Candidate Signature										

For Examiner's Use	
Examiner's Initials	
Question	Mark
1	
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TOTAL	

General Certificate of Education
Advanced Subsidiary Examination
June 2011

Mathematics MPC2

Unit Pure Core 2

Wednesday 18 May 2011 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.