June 2012 Core 4 Worked Solutions

1) a(i) $\frac{5x-6}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$ $\Rightarrow$ $5x-6 = A(x-3) + Bx$

$= (A+B)x - 3A$

equating coefficients: $5 = A+B$

$-6 = -3A$ $\Rightarrow$ $\underline{A=2}$ $\underline{B=3}$

$\frac{5x-6}{x(x-3)} = \frac{2}{x} + \frac{3}{x-3}$

(ii) $\int \frac{2}{x} + \frac{3}{x-3}\, dx = \int 2x^{-1} + 3(x-3)^{-1}\, dx$

$= 2\ln x + 3\ln(x-3) + C$

b(i) $4x^3 + 5x - 2 = (2x+1)(2x^2 + px + q) + r$

Solve either by algebraic division or by equating coefficients.

x^2: $2p + 2 = 0$ $\Rightarrow$ $\boxed{p=-1}$

x: $5x = 2qx + px$

$5 = 2q + p$

$6 = 2q$ $\Rightarrow$ $\boxed{q=3}$

constants: $-2 = q + r$

$-2 = 3 + r$ $\Rightarrow$ $\boxed{r=-5}$

(ii) $\int 2x^2 - x + 3 - \frac{5}{2x+1}\, dx$

$= \frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x - \frac{5}{2}\ln(2x+1) + C$

② (a) $\sin x - 3\cos x = R\sin(x - \alpha)$

Use compound angle formula

$$R\sin(x-\alpha) = R(\sin x\cos\alpha - \sin\alpha\cos x)$$

so $\sin x - 3\cos x = R\cos\alpha\sin x - R\sin\alpha\cos x$

equate coefficients of $\sin x$ and of $\cos x$

$$R\cos\alpha = 1$$
$$R\sin\alpha = 3$$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{1} \Rightarrow \tan\alpha = 3$$
$$\alpha = 71{\cdot}6^\circ$$

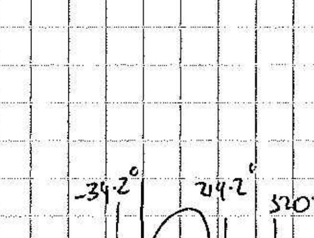
$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 3^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 10$$
$$R^2 = 10 \Rightarrow R = \sqrt{10}$$

$$\sin x - 3\cos x = \sqrt{10}\sin(x - 71{\cdot}6^\circ)$$

(b) $\sqrt{10}\sin(x - 71{\cdot}6^\circ) = -2$

$$\sin(x - 71{\cdot}6^\circ) = \frac{-2}{\sqrt{10}}$$

$$x - 71{\cdot}6^\circ = -39{\cdot}2^\circ$$
$$219{\cdot}2^\circ$$
$$320{\cdot}8^\circ$$



$$x = 32{\cdot}4^\circ,\ 290{\cdot}8^\circ$$

(3)(a) $(1+4x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(4x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(4x)^2 + \cdots$

$= 1 + 2x - 2x^2 + \cdots$

(b)(i) $(4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left[1 - \frac{1}{2}\left(\frac{-x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{4}\right)^2\right]$

$= \frac{1}{2}\left[1 + \frac{x}{8} + \frac{3}{128}x^2\right] = \frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2$

(ii) $\left|\frac{x}{4}\right| < 1 \implies |x| < 4$

(c) $\frac{\sqrt{1+4x}}{\sqrt{4-x}} = (1+4x)^{\frac{1}{2}}(4-x)^{-\frac{1}{2}}$

$\approx (1+2x-2x^2)\left(\frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2\right)$

$= \frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2$

$+ x + \frac{1}{8}x^2$ + higher powers of x

$- x^2$

$= \frac{1}{2} + \frac{17}{16}x - \frac{221}{256}x^2$

(4)(a)(i) $V = P\left(1+\frac{r}{100}\right)^n$

$r=3 \quad P=1000$

$V = 1000\left(1+\frac{3}{100}\right)^5 = 1159.27 \qquad V = £1160$

(ii) Let $V = 2000$

$2000 = 1000\left(1+\frac{3}{100}\right)^n \implies 2 = (1.03)^n$

$\log 2 = n \log(1.03)$

$n = \frac{\log 2}{\log(1.03)} = 23.45$

So after 24 years

(b) $V_B = 1000(1+0.03)^n$ $\quad V_W = 1500(1+0.015)^n$

$V_B = 1000(1.03)^n$ $\quad V_W = 1500(1.015)^n$

$$1000(1.03)^T = 1500(1.015)^T$$

$$\left(\frac{1.03}{1.015}\right)^T = \frac{3}{2}$$

$$T\log\left(\frac{1.03}{1.015}\right) = \log\left(\frac{3}{2}\right)$$

$$T = \frac{\log\left(\frac{3}{2}\right)}{\log\left(\frac{1.03}{1.015}\right)} = 27.64$$

So $T = 28$ years.

5) $x = 2\cos\theta \qquad y = 3\sin 2\theta$

a(i) $\frac{dx}{d\theta} = -2\sin\theta \qquad \frac{dy}{d\theta} = 6\cos 2\theta$

$\frac{d\theta}{dx} = \frac{-1}{2}\operatorname{cosec}\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = (6\cos 2\theta)\left(\frac{-1}{2}\operatorname{cosec}\theta\right)$$

$$= \frac{-3\cos 2\theta}{\sin\theta}$$

Double angle: $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{-3(1-2\sin^2\theta)}{\sin\theta} = \frac{-3}{\sin\theta} + \frac{6\sin^2\theta}{\sin\theta}$$

$$= -3\operatorname{cosec}\theta + 6\sin\theta$$

a(ii) gradient of tangent at $\theta = \frac{\pi}{6}$

$$6\sin\frac{\pi}{6} - \frac{3}{\sin\frac{\pi}{6}} = -3$$

gradient of normal $= \frac{1}{3}$

(b) $y = 3\sin 2\theta$

$= 3(2\sin\theta\cos\theta) = 6\sin\theta\cos\theta$ (using double-angle formula)

Remember $x = 2\cos\theta$ so we want to eliminate $\sin\theta$ from above expression.

$\sin\theta = \pm\sqrt{1-\cos^2\theta}$ using $\sin^2\theta + \cos^2\theta = 1$

$$y = \pm 6\sqrt{1-\cos^2\theta}\,\cos\theta$$
$$= \pm 6\sqrt{1-\frac{x^2}{4}}\left(\frac{x}{2}\right)$$
$$= \pm 3x\sqrt{1-\frac{x^2}{4}}$$

$$\left(\begin{aligned} x &= 2\cos\theta \\ \frac{x}{2} &= \cos\theta \\ \frac{x^2}{4} &= \cos^2\theta \end{aligned}\right)$$

$$y^2 = 9x^2\left(1-\frac{x^2}{4}\right)$$
$$y^2 = \frac{9}{4}x^2(4-x^2)$$

$$\left(1-\frac{x^2}{4} = \frac{1}{4}(4-x^2)\right)$$

6) $9x^2 - 6xy + 4y^2 = 3$

$$\frac{d}{dx}(9x^2) - \frac{d}{dx}(6xy) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(3)$$

$$\boxed{\begin{aligned} \frac{d}{dx}(6xy) &= \frac{d}{dx}(6x)\times y + \frac{d}{dy}(y)\frac{dy}{dx}\times 6x \\ &= 6y + 6x\frac{dy}{dx} \\ \frac{d}{dx}(4y^2) &= \frac{d}{dy}(4y^2)\frac{dy}{dx} = 8y\frac{dy}{dx} \end{aligned}}$$

so $18x - 6y - 6x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(8y-6x) = 6y - 18x$$
$$\frac{dy}{dx} = \frac{6y-18x}{8y-6x}$$

at stationary points $\frac{dy}{dx} = 0$

$\Rightarrow 6y = 18x \Rightarrow \boxed{y = 3x}$ →

Substitute $y = 3x$ into equation of curve.

$$9x^2 - 6x(3x) + 4(3x)^2 = 3$$
$$9x^2 - 18x^2 + 36x^2 = 3$$
$$27x^2 = 3$$
$$x = \pm\sqrt{\frac{3}{27}} = \pm\frac{1}{3}$$

$x = \frac{1}{3}$ $\quad$ $x = -\frac{1}{3}$

$y = 1$ $\quad$ $y = -1$

$\left(\frac{1}{3}, 1\right)$ $\quad$ $\left(-\frac{1}{3}, -1\right)$

7) $r_1 = \begin{bmatrix} 2\lambda \\ -2 \\ q-\lambda \end{bmatrix}$ $\quad$ $r_2 = \begin{bmatrix} 8+2\mu \\ 3+5\mu \\ 5+4\mu \end{bmatrix}$

at point P :
$$2\lambda = 8 + 2\mu \quad (1)$$
$$-2 = 3 + 5\mu \quad (2)$$
$$q - \lambda = 5 + 4\mu \quad (3)$$

(2): $-2 = 3 + 5\mu \Rightarrow 5\mu = -5$ $\quad \boxed{\mu = -1}$

(1): $2\lambda = 8 + 2(-1) = 6 \Rightarrow 2\lambda = 6$ $\quad \boxed{\lambda = 3}$

(3) $q - 3 = 5 + 4(-1) = 1 \Rightarrow \boxed{q = 4}$

$$P = (a, b, c) = (2\lambda, -2, 4-\lambda)$$
$$= (6, -2, 1)$$

$$\vec{OP} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$$

(b)(i) $\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\5\\4\end{pmatrix} = 2\times 2 + 0 + (-1)\times 4 = 4-4 = 0$

$\Rightarrow$ perpendicular.

(c) A lies on l_1 where $\lambda = 1$

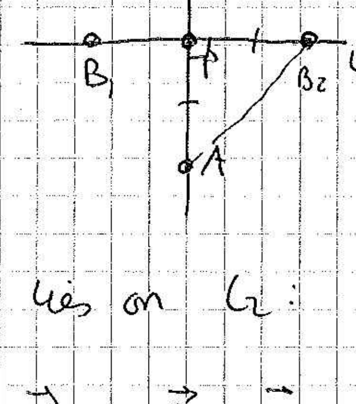
$\vec{AP} = \vec{OP} - \vec{OA}$ $\qquad \vec{OP} = \begin{pmatrix}6\\-2\\1\end{pmatrix}$

$= \begin{pmatrix}6\\-2\\1\end{pmatrix} - \begin{pmatrix}2\\-2\\3\end{pmatrix}$ $\qquad OA = \begin{pmatrix}2\lambda\\-2\\4-\lambda\end{pmatrix} = \begin{pmatrix}2\\-2\\3\end{pmatrix}$

$= \begin{pmatrix}4\\0\\-2\end{pmatrix}$

$AP^2 = 4^2 + 0^2 + (-2)^2 = 16 + 4 = 20.$

b(ii)



isosceles triangle $\Rightarrow$ $|AP| = |PB| = \sqrt{20}$

$\underline{PB^2 = AP^2 = 20}$

B lies on l_2: $\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}8+2\mu\\3+5\mu\\5+4\mu\end{pmatrix}$

$\vec{PB} = \vec{OB} - \vec{OP} = \begin{pmatrix}8+2\mu\\3+5\mu\\5+4\mu\end{pmatrix} - \begin{pmatrix}6\\-2\\1\end{pmatrix} = \begin{pmatrix}2+2\mu\\5+5\mu\\4+4\mu\end{pmatrix}$

$PB^2 = 20 \Rightarrow (2+2\mu)^2 + (5+5\mu)^2 + (4+4\mu)^2 = 20$

$\begin{matrix}4\mu^2 + 8\mu + 4\\ +\ 25\mu^2 + 50\mu + 25\\ 16\mu^2 + 32\mu + 16\end{matrix} = 20 \Rightarrow 45\mu^2 + 90\mu + 25 = 0$

$9\mu^2 + 18\mu + 5 = 0$

$(3\mu+5)(3\mu+1) = 0$

$\mu = -\frac{1}{3} \Rightarrow \vec{OB} = \begin{pmatrix}22/3\\4/3\\11/3\end{pmatrix}$ $\qquad \mu = -\frac{5}{3} \Rightarrow \vec{OB}\ \begin{pmatrix}14/3\\-16/3\\-5/3\end{pmatrix}$

(8)(a) $\frac{dh}{dt} \propto (2-h)$ $\qquad \frac{dh}{dt} = k(2-h)$

b) (i) $\frac{dx}{dt} = \frac{1}{15x\sqrt{2x-1}}$ (separate variables)

$$\int 15x\sqrt{2x-1}\,dx = \int 1\,dt$$

LHS: use integration by parts.

let $u = 15x \qquad v' = (2x-1)^{\frac{1}{2}}$

$u' = 15 \qquad v = \frac{1}{2} \times \frac{2}{3}(2x-1)^{\frac{3}{2}} = \frac{1}{3}(2x-1)^{\frac{3}{2}}$

$$uv - \int u'v = 5x(2x-1)^{\frac{3}{2}} - \int 5(2x-1)^{\frac{3}{2}}\,dx$$

$$= 5x(2x-1)^{\frac{3}{2}} - \frac{5}{2} \times \frac{2}{5}(2x-1)^{\frac{5}{2}} + C$$

$$= 5x(2x-1)^{\frac{3}{2}} - (2x-1)^{\frac{5}{2}} + C$$

so, since $\int 1\,dt = t$

$$t = 5x(2x-1)^{\frac{3}{2}} - (2x-1)^{\frac{5}{2}} + C$$

We are given that $t = 0$ when $x = 1$

$0 = 5 - 1 + C \Rightarrow \underline{C = -4}$

So $t = 5x(2x-1)^{\frac{3}{2}} - (2x-1)^{\frac{5}{2}} - 4$

(ii) $t = 10(3)^{\frac{3}{2}} - (3)^{\frac{5}{2}} - 4$

$= 32.373$

$= 32.4$ min (1dp)