Core 4 January 2012 Worked Solutions

(1)(a) $\frac{2x+3}{4x^2-1} = \frac{A}{(2x-1)} + \frac{B}{(2x+1)}$

Multiply by $(2x+1)(2x-1)$

$2x+3 = A(2x+1) + B(2x-1)$

$2x+3 = (2A+2B)x + A - B$

Equate coefficients

$2 = 2A + 2B$ (1)

$3 = A - B$ (2) $(\times 2)$

$6 = 2A - 2B$ (3)

$+ \quad 6 = 2A - 2B$ (3)

$8 = 4A$

$\boxed{A = 2}$

$\boxed{B = -1}$

So $\frac{2x+3}{4x^2-1} = \frac{2}{(2x-1)} - \frac{1}{(2x+1)}$

(b) $\frac{12x^3 - 7x - 6}{4x^2-1} = Cx + \frac{D(2x+3)}{4x^2-1}$

$\times (4x^2-1)$

$12x^3 - 7x - 6 = Cx(4x^2-1) + D(2x+3)$

$12x^3 - 7x - 6 = 4Cx^3 - Cx + 2Dx + 3D$

$= 4Cx^3 + (2D - C)x + 3D$

$12 = 4C \Rightarrow \boxed{C = 3}$ $\qquad 3D = -6 \Rightarrow \boxed{D = -2}$

$\frac{12x^3 - 7x - 6}{4x^2-1} = 3x - \frac{2(2x+3)}{4x^2-1}$

(c) $\frac{12x^3 - 7x - 6}{4x^2 - 1} = 3x - \frac{2(2x+3)}{4x^2-1}$

$= 3x - \frac{4}{2x-1} + \frac{2}{2x+1}$ Using (a) and (b)

$\int_1^2 \cdots \, dx = \int_1^2 3x - 4(2x-1)^{-1} + 2(2x+1)^{-1} \, dx$

$= \left[\frac{3}{2}x^2 - \frac{4}{2}\ln(2x-1) + \frac{2}{2}\ln(2x+1)\right]_1^2$

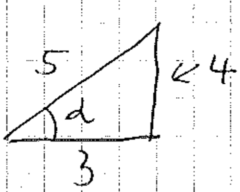
$= \left[\left(\frac{12}{2} - 2\ln 3 + \ln 5\right) - \left(\frac{3}{2} + \ln 3\right)\right]$

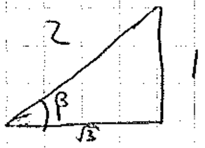
$= \frac{9}{2} - 2\ln 3 + \ln 5 - \ln 3$

$= \frac{9}{2} - 3\ln 3 + \ln 5 = \frac{9}{2} - \ln 3^3 + \ln 5$

$= \frac{9}{2} + \ln\left(\frac{5}{3^3}\right) = \frac{9}{2} + \ln\left(\frac{5}{27}\right)$

(2)



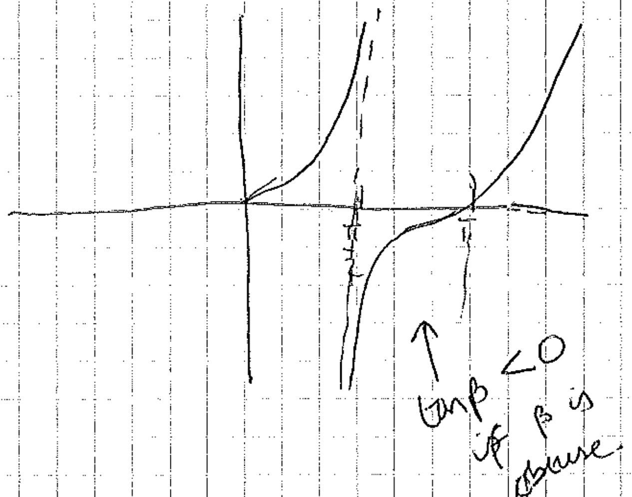




$$\sqrt{2^2 - 1^2} = \pm\sqrt{3}$$

a) (i) $\alpha$ is acute. $\tan\alpha = \frac{4}{3}$

(ii) $\beta$ is obtuse. $\tan\beta = \frac{\pm 1}{\sqrt{3}}$

$$\tan\beta = \frac{\pm 1}{\sqrt{3}}$$




(b) $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{4}{3} - \frac{1}{\sqrt{3}}}{1 - \frac{4}{3}\left(\frac{-1}{\sqrt{3}}\right)}$$

$$= \frac{\frac{4}{3} - \frac{\sqrt{3}}{3}}{1 + \frac{4}{3\sqrt{3}}} \begin{matrix} \times 3\sqrt{3} \\ \times 3\sqrt{3} \end{matrix}$$

$$= \frac{4\sqrt{3} - 3}{3\sqrt{3} + 4}$$

3(a) $(1+6x)^{\frac{2}{3}} \approx 1 + \left(\frac{2}{3}\right)6x + \frac{\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)}{2!}(6x)^2$

$= 1 + 4x - \frac{1}{9} \times 36x^2$

$= 1 + 4x - 4x^2$

(b) $(8+6x)^{\frac{2}{3}} = 8^{\frac{2}{3}}\left(1+\frac{3}{4}x\right)^{\frac{2}{3}}$

$= 4\left(1+\frac{3}{4}x\right)^{\frac{2}{3}}$

$\approx 4\left[1 + \frac{2}{3}\left(\frac{3}{4}x\right) + \frac{\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\left(\frac{3}{4}x\right)^2}{2!}\right]$

$= 4\left[1 + \frac{1}{2}x - \frac{1}{9} \times \frac{9}{16}x^2\right]$

$= 4 + 2x - \frac{1}{4}x^2$

(c) $\sqrt[3]{(8+6x)^2} \approx 4 + 2x - \frac{1}{4}x^2$

Let $8 + 6x = 10 \Rightarrow x = \frac{1}{3}$

The $\sqrt[3]{10^2} = \sqrt[3]{100} \approx 4 + 2\left(\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right)^2$

$= 4 + \frac{2}{3} - \frac{1}{36}$

$= \frac{144}{36} + \frac{24}{36} - \frac{1}{36}$

$= \frac{167}{36}.$

(5) (a) $x = 8t^2 - t$ $\quad y = \frac{3}{t} \Rightarrow t = \frac{3}{y}$

$x = 8\left(\frac{3}{y}\right)^2 - \frac{3}{y}$

$x = \frac{72}{y^2} - \frac{3}{y}$ $\quad \Rightarrow \times$ by $y^2$

$xy^2 = 72 - 3y$

$xy^2 + 3y = 72$

$k = 72$

(b)(i) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dt} = -3t^{-2}$ $\quad \frac{dx}{dt} = 16t - 1$ $\quad \Rightarrow \frac{dt}{dx} = \frac{1}{16t-1}$

$\frac{dy}{dx} = \frac{-3}{t^2(16t-1)}$

$\boxed{t = \frac{1}{4}}$ $\quad x = 8\left(\frac{1}{16}\right) - \frac{1}{4} = \frac{1}{4}$ $\quad y = \frac{3}{\frac{1}{4}} = 12$

$\frac{dy}{dx} = \frac{-3}{\frac{1}{4}(3)} = -16$

$y - 12 = -16\left(x - \frac{1}{4}\right)$ $\quad \Rightarrow y - 12 = -16x + 4$

$\boxed{y = -16x + 16}$

(ii) $x = \frac{3}{2}$ $\quad y = -16\left(\frac{3}{2}\right) + 16 = -24 + 16 = -8$

Check to see if $\left(\frac{3}{2}, -8\right)$ satisfies $xy^2 + 3y = 72$

$\left(\frac{3}{2}\right)(-8)^2 + 3(-8) = 96 - 24$

$= 72$ ✓

6 (a) $f\left(\frac{3}{4}\right) = 16\left(\frac{3}{4}\right)^2 + 11\left(\frac{3}{4}\right) - 15$

$= \frac{27}{4} + \frac{33}{4} - \frac{60}{4} = 0 \Rightarrow (4x-3)$ is a factor.

(b) $x = \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

(double-angle)

$27\cos\theta(2\cos^2\theta - 1) + 19\sin\theta(2\sin\theta\cos\theta) - 15$

$54\cos^3\theta - 27\cos\theta + 38\sin^2\theta\cos\theta - 15$

$54\cos^3\theta - 27\cos\theta + 38\cos\theta(1-\cos^2\theta) - 15 = 0$

$54\cos^3\theta - 27\cos\theta + 38\cos\theta - 38\cos^3\theta - 15 = 0$

$16\cos^3\theta + 11\cos\theta - 15 = 0$

$16x^3 + 11x - 15 = 0.$

(c)

$$
\begin{array}{r|l}
 & 4x^2 + 3x + 5 \\
\hline
4x-3 & 16x^3 + 0x^2 + 11x - 15 \\
 & 16x^3 - 12x^2 \\
\hline
 & 12x^2 + 11x - 15 \\
 - & 12x^2 - 9x \\
\hline
 & 20x - 15 \\
 - & 20x - 15 \\
\hline
 & 0
\end{array}
$$

$\Rightarrow 16x^3 + 11x - 15 = (4x-3)(4x^2+3x+5)$

$16x^3 + 11x - 15 = 0$

when $4x - 3 = 0 \Rightarrow x = \frac{3}{4}$

and $4x^2 + 3x + 5 = 0$

consider discriminant:

$b^2 - 4ac$

$9 - 4(4)(5) < 0$

$\Rightarrow$ no solutions

Only solution when $\cos\theta = \frac{3}{4}$.

(7) $\frac{dy}{dx} = y^2 x \sin 3x \qquad y = 1,\ x = \frac{\pi}{6}$

Separating variables gives:

$\int y^{-2}\,dy = \int x \sin 3x \, dx \qquad = \frac{-1}{3} x \cos 3x + \frac{1}{3}\int \cos 3x \, dx$

Integrate RHS by parts: $\qquad u = x \qquad v' = \sin 3x$

$\qquad u' = 1 \qquad v = \frac{-1}{3}\cos 3x$

$\int x \sin 3x = \frac{-1}{3} x \cos 3x + \frac{1}{9}\sin 3x + C$

$\Rightarrow -y^{-1} = \frac{-1}{3} x \cos 3x + \frac{1}{9}\sin 3x + C$

use fact that $y = 1$ when $x = \frac{\pi}{6}$:

$-1 = \frac{-1}{3}\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{2}\right) + \frac{1}{9}\sin\left(\frac{\pi}{2}\right) + C$

$-1 = \frac{1}{9} + C \Rightarrow C = -\frac{10}{9}$

$-\frac{1}{y} = \frac{-1}{3} x \cos 3x + \frac{1}{9}\sin 3x - \frac{10}{9}$

$= \frac{-3x\cos 3x}{9} + \frac{\sin 3x}{9} - \frac{10}{9}$

$\frac{-1}{y} = \frac{-3x\cos 3x + \sin 3x - 10}{9}$

$\Rightarrow y = \frac{9}{3x\cos 3x - \sin 3x + 10}$

⑧ $A = (4, -2, 3)$ $B = (2, 0, -1)$

$$r = \begin{pmatrix} 4 + \lambda \\ -2 + 5\lambda \\ 3 - 2\lambda \end{pmatrix}$$

(a)(i) $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$

(ii) $\begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = |(-2, 2, -4)| \, |(1, 5, -2)| \cos\theta$

$$-2 + 10 + 8 = \sqrt{24} \times \sqrt{30} \cos\theta$$

$$16 = \sqrt{720} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{16}{\sqrt{720}}\right) \Rightarrow \theta = 53.3457$$

$$\theta = 53^\circ$$

(b) $C: \begin{pmatrix} 4 + p \\ -2 + 5p \\ 3 - 2p \end{pmatrix}$ for some value $p$ since $C$ lies on $r$.

$\vec{AB} \cdot \vec{BC} = 0$ (since they are perpendicular).

$\vec{BC} = \begin{pmatrix} 2 + p \\ 5p - 2 \\ 4 - 2p \end{pmatrix}$ $\left( BC = \begin{pmatrix} 4 + p - 2 \\ -2 + 5p - 0 \\ 3 - 2p + 1 \end{pmatrix} = \begin{pmatrix} 2 + p \\ 5p - 2 \\ 4 - 2p \end{pmatrix} \right)$

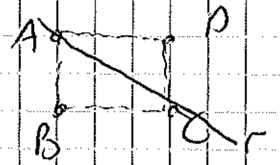
$\vec{AB} \cdot \vec{BC} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} \begin{pmatrix} 2 + p \\ 5p - 2 \\ 4 - 2p \end{pmatrix} = -2(2 + p) + 2(5p - 2) - 4(4 - 2p) = 0$

$$\Rightarrow \begin{matrix} -4 - 2p \\ -4 + 10p \\ -16 + 8p \end{matrix} = 0$$

$$-24 + 16p = 0$$

$$16p = 24 \Rightarrow p = \frac{3}{2}$$

$$\vec{BC} = \begin{pmatrix} 3.5 \\ 5.5 \\ 1 \end{pmatrix}$$




$$\vec{OD} = \vec{OA} + \vec{BC}$$ (Since $\vec{BC}$ is parallel to $\vec{AD}$) + same length

$$\vec{OD} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3.5 \\ 5.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 3.5 \\ 4 \end{pmatrix}$$

D is point $(7.5, 3.5, 4)$