

① (a) $f(x) = 4x^3 - 13x + 6$

$$f(-2) = 4(-2)^3 - 13(-2) + 6$$

$$= -32 + 26 + 6 = 0$$

(Therefore $(x+2)$ is a factor).

(b) let $x = \frac{3}{2}$

$$f\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^3 - 13\left(\frac{3}{2}\right) + 6$$

$$= \frac{27}{2} - \frac{39}{2} + \frac{12}{2} = \frac{39}{2} - \frac{39}{2} = 0$$

Therefore $(2x-3)$ is a factor.

(c) $\frac{2x^2 + x - 6}{4x^3 - 13x + 6}$

$$2x^2 + x - 6 = (2x-3)(x+2)$$

$$4x^3 - 13x + 6 = (x+2)(2x-3)(ax+b)$$

equating coefficients of x^3 and constants

$$4 = 1 \times 2 \times a \Rightarrow \underline{a = 2}$$

$$6 = 2 \times -3 \times b \Rightarrow \underline{b = -1}$$

$$\frac{2x^2 + x - 6}{4x^3 - 13x + 6} = \frac{\cancel{(2x-3)}\cancel{(x+2)}}{\cancel{(x+2)}\cancel{(2x-3)}(2x-1)} = \frac{1}{2x-1}$$

② When $t = 0$ $P = 80$

a(i) so $A = 80$

(ii) $2000 = 80k^{25}$

$$\frac{2000}{80} = k^{25} \Rightarrow 25 = k^{25}$$

$$\sqrt[25]{25} = k$$

$$k = 1.137411$$

(b) $100000 = 80(1.137411)^t$

$$1250 = 1.137411^t$$

$$\ln(1250) = t \ln(1.137411)$$

$$t = \frac{\ln(1250)}{\ln(1.137411)} = 55.38$$

so 56 years $\Rightarrow$ 2016

③ (a) (i) $(1-x)^{\frac{1}{3}} \approx 1 + \left(\frac{1}{3}\right)(-x) + \frac{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2!}(-x)^2$

$$= 1 - \frac{1}{3}x - \frac{1}{9}x^2$$

(ii) $(125 - 27x)^{\frac{1}{3}}$

$$= \left[125 \left(1 - \frac{27}{125}x\right)\right]^{\frac{1}{3}} = 5 \left(1 - \frac{27}{125}x\right)^{\frac{1}{3}}$$

$$\approx 5 \left[1 - \frac{1}{3} \left(\frac{27}{125}\right)x - \frac{1}{9} \left(\frac{27}{125}\right)^2 x^2\right]$$

$$= 5 - \frac{9}{25}x - \frac{81}{3125}x^2$$

(b) $\sqrt[3]{119} = 119^{\frac{1}{3}}$ so let $125 - 27x = 119$

$$\Rightarrow 27x = 6$$

$$x = \frac{6}{27} = \frac{2}{9}$$

$$5 - \frac{9}{25} \left(\frac{2}{9}\right) - \frac{81}{3125} \left(\frac{2}{9}\right)^2 = 4.91872$$

$$(4) (a) \quad x = 3 \cos 2\theta \quad y = 2 \cos \theta$$

$$(i) \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{d\theta} = -2 \sin \theta$$

$$\frac{dx}{d\theta} = -6 \sin 2\theta \Rightarrow \frac{d\theta}{dx} = \frac{-1}{6 \sin 2\theta}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (\text{double angle formula})$$

$$\frac{d\theta}{dx} = \frac{-1}{6 \sin 2\theta} = \frac{-1}{12 \sin \theta \cos \theta}$$

$$\frac{dy}{dx} = -2 \sin \theta \times \frac{-1}{12 \sin \theta \cos \theta} = \frac{1}{6} + \frac{1}{6 \cos \theta} = \frac{1}{6 \cos \theta}$$

$$(ii) \text{ Gradient of tangent at } \theta = \frac{\pi}{3} \text{ is } \frac{1}{6 \cos \frac{\pi}{3}} = \frac{1}{3}$$

$$\text{so gradient of normal at this point is } \boxed{-3}$$

$$\text{When } \theta = \frac{\pi}{3} \quad x = 3 \cos \left(\frac{2\pi}{3} \right) \quad y = 2 \cos \left(\frac{\pi}{3} \right)$$

$$= -\frac{3}{2} \quad = \sqrt{3}$$

$$(y - \sqrt{3}) = -3 \left(x + \frac{3}{2} \right)$$

$$(b) \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 - \cos 2x \, dx$$

$$(\text{using double angle formula: } \cos 2x = \cos^2 x - \sin^2 x)$$

$$= 1 - 2 \sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2} \left[\left(\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) - \left(-\frac{\pi}{4} - \frac{1}{2} \sin \left(-\frac{\pi}{2} \right) \right) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - \frac{1}{2} - \left(-\frac{1}{2} \right) \right] = \frac{\pi}{4} - \frac{1}{2}$$

$$5(a) \quad \vec{OA} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \quad \vec{OB} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

$$S = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

$$b(i) \quad r = \begin{pmatrix} -8+5\mu \\ 5 \\ -6-2\mu \end{pmatrix} \quad s = \begin{pmatrix} 5-\lambda \\ 1-2\lambda \\ -2+5\lambda \end{pmatrix}$$

At point of intersection:

$$5 = 1 - 2\lambda$$

$$2\lambda = -4$$

$$\boxed{\lambda = -2}$$

$$-8 + 5\mu = 5 - \lambda$$

$$-8 + 5\mu = 7$$

$$5\mu = 15$$

$$\boxed{\mu = 3}$$

Check 3rd equation:

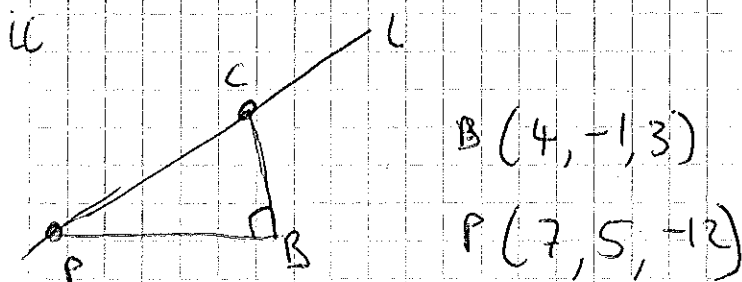
$$-6 - 2\mu = -6 - 6 = -12$$

$$-2 + 5\lambda = -2 - 10 = -12$$

✓ so lines intersect.
when $\lambda = -2$ $\mu = 3$

$$\Rightarrow \begin{pmatrix} -8+5\mu \\ 5 \\ -6-2\mu \end{pmatrix} = \begin{pmatrix} -8+15 \\ 5 \\ -6-6 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -12 \end{pmatrix}$$

$$P = (7, 5, -12)$$



$$\vec{BP} \cdot \vec{CB} = 0 \quad (\text{since perpendicular})$$

$$\vec{BP} = \vec{OP} - \vec{OB} = \begin{pmatrix} 3 \\ 6 \\ -15 \end{pmatrix}$$

Let $\vec{OC} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Then $\vec{CB} = \vec{OB} - \vec{OC} = \begin{pmatrix} 4-a \\ -1-b \\ 3-c \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 6 \\ -15 \end{pmatrix} \cdot \begin{pmatrix} 4-a \\ -1-b \\ 3-c \end{pmatrix} = 0$$

C lies on L so $a = -8 + 5\mu$ $b = 5$ $c = -6 - 2\mu$

$$\begin{pmatrix} 3 \\ 6 \\ -15 \end{pmatrix} \cdot \begin{pmatrix} 4 + 8 - 5\mu \\ -6 \\ 3 + 6 + 2\mu \end{pmatrix} = 0$$

$$3(12 - 5\mu) + 6(-6) + (-15)(9 + 2\mu) = 0$$

$$\Rightarrow \cancel{36} - 15\mu - \cancel{36} - 135 - 30\mu = 0$$

$$45\mu = 135$$

$$\boxed{\mu = -3}$$

$$\vec{OC} = \begin{pmatrix} -8 - 15 \\ 5 \\ -6 - 6 \end{pmatrix} = \begin{pmatrix} -23 \\ 5 \\ 0 \end{pmatrix}$$

C is point $(-23, 5, 0)$.

$$\textcircled{a} \quad 2y + e^{2x} y^2 = x^2 + C$$

$$x=1 \quad y=\frac{1}{e}$$

$$\frac{2}{e} + e^2 \left(\frac{1}{e}\right)^2 = 1 + C$$

$$\frac{2}{e} + 1 = 1 + C \Rightarrow C = \frac{2}{e}$$

$$(b) \quad \frac{dy}{dx}$$

$$\frac{d}{dx} 2y + \frac{d}{dx} e^{2x} y^2 = \frac{d}{dx} x^2 + \frac{d}{dx} 2e^{-1}$$

$$\frac{d}{dx} 2y = \frac{d}{dy} 2y \times \frac{dy}{dx} = 2 \frac{dy}{dx}$$

$$\begin{aligned} \frac{d}{dx} e^{2x} y^2 &= e^{2x} \frac{d}{dy} y^2 \times \frac{dy}{dx} + y^2 \frac{d}{dx} e^{2x} \quad (\text{using product rule}) \\ &= 2y e^{2x} \frac{dy}{dx} + 2y^2 e^{2x} \end{aligned}$$

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx} 2e^{-1} = 0$$

$$2 \frac{dy}{dx} + 2y e^{2x} \frac{dy}{dx} + 2y^2 e^{2x} = 2x$$

$$(2ye^{2x} + 2) \frac{dy}{dx} = 2x - 2y^2 e^{2x}$$

$$\frac{dy}{dx} = \frac{2x - 2y^2 e^{2x}}{2ye^{2x} + 2} = \frac{x - y^2 e^{2x}}{ye^{2x} + 1}$$

(c) Stationary point $\Rightarrow \frac{dy}{dx} = 0$

$$\frac{x - y^2 e^{2x}}{y e^{2x} + 1}$$

evaluated at $x=1$
 $y = \frac{1}{e} = e^{-1}$

gives:

$$\frac{1 - e^{-2} e^2}{e^2 e^{-1} + 1} = \frac{1-1}{e+1} = 0 \quad \checkmark$$

(7) (a) $\frac{dA}{dt} = -k$

When $t=0$ $A = 4\pi \times 60^2 = 14400\pi$
 $t=9$ $A = 4\pi \times 30^2 = 3600\pi$

Separate variables in (a)

$$\int dA = \int -k dt$$

$$\Rightarrow \boxed{A = -kt + C}$$

$$14400\pi = 0 + C \Rightarrow C = 14400\pi$$

$$3600\pi = -9k + 14400\pi$$

$$9k = 10800\pi$$

$$k = 1200\pi$$

$$\boxed{A = -1200\pi t + 14400\pi}$$

$$A = -1200\pi(t - 12)$$

$$\boxed{A = 1200\pi(12 - t)}$$

(ii) $A=0 \Rightarrow 0 = 1200\pi(12 - t)$

$$\Rightarrow t = 12$$

12 days

$$\textcircled{8} \quad \frac{1}{(3-2x)(1-x)^2} = \frac{A}{(3-2x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

$$\times (3-2x)(1-x)^2$$

$$1 = A(1-x)^2 + B(3-2x)(1-x) + C(3-2x)$$

$$\text{let } \underline{x=1} \Rightarrow 1 = 0 + 0 + C(3-2) \\ \Rightarrow 1 = C$$

$$\boxed{C=1}$$

$$\text{let } \underline{x = \frac{3}{2}} \Rightarrow 1 = A\left(\frac{-1}{2}\right)^2 + 0 + 0 \\ 1 = \frac{A}{4}$$

$$\boxed{A=4}$$

To find B you can now use any value for x,
so use $x=0$.

$$1 = 4(1)^2 + B(3)(1) + 3$$

$$1 = 7 + 3B$$

$$3B = -6 \Rightarrow \boxed{B = -2}$$

$$\frac{1}{(3-2x)(1-x)^2} = \frac{4}{(3-2x)} - \frac{2}{(1-x)} + \frac{1}{(1-x)^2}$$

(b) $\frac{dy}{dx} = \frac{2\sqrt{y}}{(3-2x)(1-x)^2}$ Separating variables gives:

$$\int \frac{1}{2\sqrt{y}} dy = \int \frac{1}{(3-2x)(1-x)^2} dx$$

$$\frac{1}{2} \int y^{-\frac{1}{2}} dy = \int \left[4(3-2x)^{-1} - 2(1-x)^{-1} + (1-x)^{-2} \right] dx \quad (\text{using (a)})$$

$$\frac{1}{2} [2y^{\frac{1}{2}}] = 4 \times \frac{1}{2} \ln(3-2x) + 2 \ln(1-x) + \frac{1}{1-x} + C$$

$$\sqrt{y} = -2 \ln(3-2x) + 2 \ln(1-x) + \frac{1}{1-x} + C$$

use fact that $y=0$ when $x=0$ to find C .

$$0 = -2 \ln 3 + 1 + C$$

$$C = 2 \ln 3 - 1$$

so $\boxed{\sqrt{y} = -2 \ln(3-2x) + 2 \ln(1-x) + \frac{1}{1-x} + 2 \ln 3 - 1}$

Simplify RHS:

consider first $\frac{1}{1-x} - 1 = \frac{1}{1-x} - \frac{1-x}{1-x} = \frac{1-1+x}{1-x} = \frac{x}{1-x}$

then:

$$-2 \ln(3-2x) + 2 \ln(1-x) + 2 \ln 3$$

$$= \ln(3-2x)^{-2} + \ln(1-x)^2 + \ln 3^2$$

$$= \ln \left[\frac{3^2 (1-x)^2}{(3-2x)^2} \right] = \ln \left[\frac{3(1-x)}{(3-2x)} \right]^2 = 2 \ln \left[\frac{3-3x}{3-2x} \right]$$

$\Rightarrow \boxed{y^{\frac{1}{2}} = 2 \ln \left[\frac{3-3x}{3-2x} \right] + \frac{x}{1-x}}$