Core 4 January 2011

1)(a) $2\sin x + 5\cos x = R\sin(x+\alpha)$

$\sin(x+\alpha) = \sin x\cos\alpha + \sin\alpha\cos x$

$\Rightarrow 2\sin x + 5\cos x = R\cos\alpha\sin x + R\sin\alpha\cos x$

equating coefficients gives $2 = R\cos\alpha$ $5 = R\sin\alpha$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 2^2+5^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 29$ $\Rightarrow R^2 = 29$ $R = \sqrt{29}$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{5}{2}$ $\Rightarrow \tan\alpha = \frac{5}{2}$

$\alpha = 68.2°$

$\sqrt{29}\sin(x + 68.2°)$

b)(i) Max value of $\sin\theta = 1$ so max value is $\sqrt{29}$.

(ii) $\sin(x+68.2) = 1 \Rightarrow x + 68.2 = 90°$

$x = 21.8°$

2) a(i) $f\left(\frac{-1}{3}\right) = 9\left(\frac{-1}{3}\right)^3 + 18\left(\frac{-1}{3}\right)^2 - \left(\frac{-1}{3}\right) - 2$

$= \frac{-1}{3} + 2 + \frac{1}{3} - 2 = 0$

$\therefore (3x+1)$ is a factor.

(ii) either by equating coefficients or:

$$\begin{array}{r} 3x^2+5x-2 \\ 3x+1 \overline{\big)\, 9x^3+18x^2-x-2} \\ -\ \ 9x^3+3x^2 \qquad\quad \\ \hline 15x^2-x-2 \\ -\ \ 15x^2+5x \quad\ \ \\ \hline -6x-2 \\ -\ \ -6x-2 \\ \hline 0 \end{array}$$

$3x^2+5x-2 = (3x-1)(x+2)$

$\Rightarrow f(x) = (3x+1)(3x-1)(x+2)$

ii) $9x^3 + 21x^2 + 6x = 3x(3x^2 + 7x + 2)$

$= 3x(3x+1)(x+2)$

$$\frac{9x^3 + 21x^2 + 6x}{9x^3 + 18x^2 - x - 2} = \frac{3x\cancel{(3x+1)}\cancel{(x+2)}}{\cancel{(3x+1)}(3x-1)\cancel{(x+2)}} = \frac{3x}{3x-1}$$

b) $f\left(\frac{2}{3}\right) = -4$

$9\left(\frac{2}{3}\right)^3 + p\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right) - 2 = -4$

$\frac{72}{27} - \frac{18}{27} + p\left(\frac{4}{9}\right) = -2$

$\frac{4}{9}p = -4 \quad \Rightarrow \underline{p = -9}$

3 (a) $$\frac{A(3+5x) + B(1+x)}{(1+x)(3+5x)}$$

$3 + 9x = A(3+5x) + B(1+x)$

$= 3A + 5Ax$

$+ B + Bx$

$\Rightarrow \quad 3 = 3A + B$ (1) and $9 = 5A + B$ (2)

(2) − (1) gives $6 = 2A$

$\Rightarrow A = 3, \quad B = -6.$

(b) $\frac{3}{1+x} = 3(1+x)^{-1} \simeq 3\left(1 + (-1)x + \frac{(-1)(-2)x^2}{2!}\right)$

$= 3(1 - x + x^2)$

$\frac{-6}{(3+5x)} = -6(3+5x)^{-1} = -6\left[3^{-1}\left(1+\frac{5}{3}x\right)^{-1}\right] = -2\left(1+\frac{5}{3}x\right)^{-1}$

$= -2\left(1 + (-1)\left(\frac{5}{3}x\right) + \frac{(-1)(-2)\left(\frac{5}{3}x\right)^2}{2!}\right) = -2\left(1 - \frac{5}{3}x + \frac{25}{9}x^2\right)$

so $\frac{3}{1+x} + \frac{-6}{3+5x} \simeq 3(1-x+x^2) - 2\left(1 - \frac{5}{3}x + \frac{25x^2}{9}\right)$

$= 3 - 3x + 3x^2$

$\quad -2 + \frac{10}{3}x - \frac{50}{9}x^2 \quad = 1 + \frac{1}{3}x - \frac{23}{9}x^2$

(c) $\left|\frac{5}{3}x\right| < 1 \qquad |x| < \frac{3}{5}.$

4) $x = 3e^t \quad y = e^{2t} - e^{-2t}$

(a)(i) $\frac{dx}{dt} = 3e^t \quad \left(\frac{dt}{dx} = \frac{1}{3}e^{-t}\right) \qquad \frac{dy}{dt} = 2e^{2t} + 2e^{-2t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{3}e^{-t}\left(2e^{2t} + 2e^{-2t}\right)$

$= \frac{2}{3}\left(e^{t} + e^{-t}\right)$

$\underline{x=0}$: $\frac{dy}{dx} = \frac{2}{3}(2) = \frac{4}{3}$

(ii) $t = 0$: $x = 3 \qquad y = 1 - 1 = 0$

$y - 0 = \frac{4}{3}(x-3)$

$y = \frac{4}{3}(x-3)$

(b) $y = e^{2t} - e^{-2t} \qquad x = 3e^t$

$x^2 = 9e^{2t}$

$\Rightarrow e^{2t} = \frac{x^2}{9}$ and $e^{-2t} = \frac{9}{x^2}$

$\Rightarrow y = \frac{x^2}{9} - \frac{9}{x^2} \qquad \underline{(k=9)}$

5) (a) $M_0 = 10 \quad t = 14$

$M = 10 \times 2^{-\frac{14}{8}} = 2.97 \quad \Rightarrow \quad 3$ grams

b) $\frac{\cancel{M_0}}{16} = \cancel{M_0} \times 2^{-\frac{d}{8}} \quad \Rightarrow \quad \frac{1}{16} = 2^{-\frac{d}{8}}$

$2^{-4} = \frac{1}{16}$ so: $\frac{-d}{8} = -4 \quad \Rightarrow \quad d = 4 \times 8 = 32$ days

(c) $\frac{1}{100} M_0 = M_0 \times 2^{-\frac{n}{8}}$

$\frac{1}{100} = 2^{-\frac{n}{8}} \quad \Rightarrow \quad \ln\left(\frac{1}{100}\right) = \ln\left(2^{-\frac{n}{8}}\right)$

$-\ln 100 = \frac{-n}{8} \ln 2$

so $n = \frac{8 \ln 100}{\ln 2} = 53.15$

so after $\underline{54 \text{ days}}$.

6) (a)(i) use $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\frac{2\tan x}{1-\tan^2 x} + \tan x = 0 \rightarrow \frac{2\tan x}{1-\tan^2 x} + \frac{(1-\tan^2 x)\tan x}{1-\tan^2 x} = 0$

$\frac{2\tan x + (1-\tan^2 x)\tan x}{(1-\tan^2 x)} = 0$

$\tan x\left(2 + 1 - \tan^2 x\right) = 0$

$\tan x(3 - \tan^2 x) = 0$

$\underline{\tan x = 0} \qquad 3 - \tan^2 x = 0$

$\underline{\tan^2 x = 3}$

i) $\tan x = 0$

$x = \cancel{0°}, \cancel{180°}$

(not in interval)

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

$x = 60°, \cancel{240°}$

$\cancel{-60°}, 120°$

$x = 60°, 120°$

b)(i) $\sin 2x = \cos x \cos 2x$

Use: $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$\Rightarrow 2\sin x \cos x = \cos x(1 - 2\sin^2 x)$

$\cos x \neq 0$ so can divide by $\cos x$

$2\sin x = 1 - 2\sin^2 x$

$2\sin^2 x + 2\sin x - 1 = 0$ as required.

(ii) let $X = \sin x$

$\Rightarrow 2X^2 + 2X - 1 = 0$ $a=2$ $b=2$ $c=-1$

$X = \dfrac{-2 \pm \sqrt{4 - 4(2)(-1)}}{4} = \dfrac{-2 \pm \sqrt{12}}{4} = -\dfrac{1}{2} \pm \dfrac{2\sqrt{3}}{4}$

so $X = \dfrac{-1 \pm \sqrt{3}}{2}$

$\boxed{\sin x = \dfrac{\sqrt{3} - 1}{2}}$

$\cancel{\sin x = \dfrac{-\sqrt{3} - 1}{2}}$

no solution.

7(a)(i) $\frac{dx}{dt} = \sqrt{x}\,\sin\left(\frac{t}{2}\right)$ Separate variables;

$$\int x^{-\frac{1}{2}}\,dx = \int \sin\left(\frac{t}{2}\right) dt$$

$$2x^{\frac{1}{2}} = -2\cos\left(\frac{t}{2}\right) + k$$

$$\sqrt{x} = c - \cos\left(\frac{t}{2}\right) \quad (\text{where } c = \frac{k}{2})$$

$$\underline{x = \left(c - \cos\left(\frac{t}{2}\right)\right)^2}$$

(ii) $x = 1,\ t = 0$

$$1 = (c - \cos(0))^2 = (c-1)^2$$

$\Rightarrow$ $c - 1 = 1$ or $c - 1 = -1$

$c = 2$ $\quad$ $c = 0$

$\downarrow$ $\quad$ $\downarrow$

$\underline{x = \left(2 - \cos\left(\frac{t}{2}\right)\right)^2}$ $\quad$ $x = \cos^2\left(\frac{t}{2}\right)$

$$\boxed{\begin{aligned} a &= 2 \\ b &= \tfrac{1}{2} \end{aligned}}$$

(b) $\frac{dx}{dt} = \sqrt{x}\,\sin\left(\frac{t}{2}\right)$ $\quad x = 1 \quad t = 0$ (as in (a)).

(i) $x = \left(2 - \cos\left(\frac{1}{2}t\right)\right)^2$ so x is max at max value of $2 - \cos\left(\frac{1}{2}t\right)$

$$= 2 - (-1) = 3$$

$x_{max} = 3^2 = 9\text{m}$

(ii) $5 = \left(2 - \cos\left(\frac{t}{2}\right)\right)^2 \Rightarrow \sqrt{5} = 2 - \cos\left(\frac{t}{2}\right)$

$\cos\left(\frac{t}{2}\right) = 2 - \sqrt{5}$ so $\frac{1}{2}t = 1 \cdot 809$

$t = 3 \cdot 618 \rightarrow t = \underline{3 \cdot 6\text{s}}$

8) (a)(i) $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 6 - 2 - 3 = 1$

$\sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$

$\sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$

$\Rightarrow \quad 1 = \sqrt{14} \times \sqrt{14} \times \cos\theta \qquad (A \cdot B = |A||B| \cos\theta)$

$\frac{1}{14} = \cos\theta$

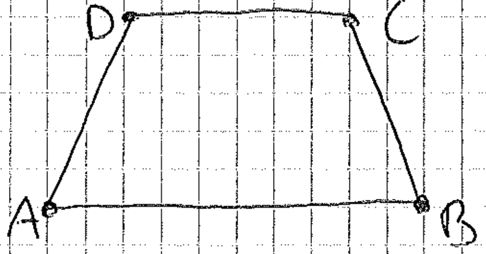
$\Rightarrow \theta = 85.9^\circ$

(b)(i) $L_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$D = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 10 \end{pmatrix}$

$\Rightarrow \quad L_2 = \begin{pmatrix} 7 \\ -4 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

(ii)



$|AD| = |BC|$

and C lies on L_2.

so for some value of $\mu = t$

$C = \begin{pmatrix} 7 + 3t \\ -4 + 2t \\ 10 - t \end{pmatrix}$

$|\vec{AD}| = \left| \begin{pmatrix} 7-3 \\ -4+2 \\ 10-4 \end{pmatrix} \right| = \left| \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} \right| = \sqrt{16+4+36} = \sqrt{56}$

) $|CB| = \sqrt{56}$

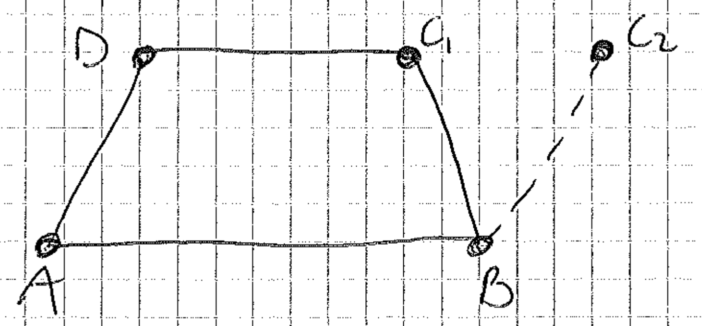
$$\vec{CB} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 7+3t \\ -4+2t \\ 10-t \end{pmatrix} = \begin{pmatrix} -1-3t \\ 4-2t \\ -7+t \end{pmatrix}$$

$$(-1-3t)^2 = 1+6t+9t^2$$
$$(4-2t)^2 = 16-16t+4t^2$$
$$(-7+t)^2 = 49-14t+t^2$$

$$|CB| = \sqrt{66-24t+14t^2} = \sqrt{56}$$

$$14t^2-24t+66 = 56$$
$$14t^2-24t+10 = 0$$
$$7t^2-12t+5 = 0$$
$$(7t-5)(t-1)$$
$$t = \frac{5}{7} \qquad t = 1$$

So 2 possibilities for C such that C lies on l_2 and $|CB| = |AD|$



The point we want is the one closer to D.

$D(7, -4, 10)$

$$C = \begin{pmatrix} 10 \\ -2 \\ 9 \end{pmatrix} \quad (t=1)$$

$$\boxed{C = \begin{pmatrix} 9\frac{1}{7} \\ -2\frac{4}{7} \\ 9\frac{2}{7} \end{pmatrix} \quad \left(t = \frac{5}{7}\right)}$$