

June 2010 Core 4

$$\begin{aligned} \textcircled{1} \text{ (a)} \quad f\left(\frac{1}{4}\right) &= 8\left(\frac{1}{4}\right)^3 + 6\left(\frac{1}{4}\right)^2 - 14\left(\frac{1}{4}\right) - 1 \\ &= \frac{8}{64} + \frac{6}{16} - \frac{14}{4} - 1 \\ &= \frac{1}{8} + \frac{3}{8} - \frac{28}{8} - \frac{8}{8} = -\frac{32}{8} = -4 \end{aligned}$$

$$\text{(b) (i)} \quad g(x) = 8x^3 + 6x^2 - 14x + d$$

$$(4x-1) \text{ is a factor} \Rightarrow g\left(\frac{1}{4}\right) = 0$$

$$g\left(\frac{1}{4}\right) = \frac{-24}{8} + d$$

$$\Rightarrow -3 + d = 0$$

$$\underline{\underline{d=3}}$$

(ii) Comparing coefficients or:

$$\begin{array}{r} 2x^2 + 2x - 3 \\ 4x-1 \overline{) 8x^3 + 6x^2 - 14x + 3} \\ \underline{-8x^2 - 2x} \phantom{+ 3} \\ -8x^2 - 14x + 3 \\ \underline{8x^2 - 2x} \phantom{+ 3} \\ -12x + 3 \\ \underline{-12x + 3} \\ 0 \end{array}$$

$$a=2$$

$$b=2$$

$$c=-3$$

$$\textcircled{2} \text{ (a)} \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dt} = 6t$$

$$\frac{dx}{dt} = -3 \Rightarrow \frac{dt}{dx} = -\frac{1}{3}$$

$$\frac{dy}{dx} = \frac{-6t}{3} = -2t$$

$$\text{(b)} \quad \underline{t=1} \quad x = 1-3 = -2 \quad y = 1+2 = 3$$

$$m = -2 \quad \text{(tangent)}$$

$$m_{\text{normal}} = \frac{1}{2}$$

$$\Rightarrow \underline{\underline{y-3 = \frac{1}{2}(x+2)}}$$

$$\begin{aligned}
 2) \quad x &= 1 - 3t \\
 3t &= 1 - x \\
 t &= \frac{1-x}{3} \quad \Rightarrow \quad y = 1 + 2t^3 \\
 &= 1 + 2\left(\frac{1-x}{3}\right)^3 \\
 y &= 1 + \frac{2}{27}(1-x)^3
 \end{aligned}$$

$$\begin{aligned}
 3) \quad \frac{7x-3}{(x+1)(3x-2)} &= \frac{A}{(x+1)} + \frac{B}{(3x-2)} \\
 &= \frac{A(3x-2) + B(x+1)}{(x+1)(3x-2)}
 \end{aligned}$$

$$7x-3 = A(3x-2) + B(x+1)$$

$$\underline{x=-1} \quad -10 = A(-5) \quad \Rightarrow \quad \underline{A=2}$$

$$\begin{aligned}
 \underline{x=\frac{2}{3}} \quad 7\left(\frac{2}{3}\right) - 3 &= B\left(\frac{5}{3}\right) \\
 \frac{14}{3} - \frac{9}{3} &= B\left(\frac{5}{3}\right) \\
 \frac{5}{3} &= \frac{5}{3}B \quad \Rightarrow \quad \underline{B=1}
 \end{aligned}$$

$$(ii) \quad \int \frac{2}{(x+1)} + \frac{1}{(3x-2)} dx = 2 \ln|x+1| + \frac{1}{3} \ln|3x-2| + C$$

$$(b) \quad P + \frac{Qx+R}{2x^2-x+1} = \frac{P(2x^2-x+1)}{(2x^2-x+1)} + \frac{Qx+R}{(2x^2-x+1)}$$

$$\begin{aligned}
 6x^2 + x + 2 &= P(2x^2 - x + 1) + Qx + R \\
 &= 2Px^2 + (Q-P)x + (P+R)
 \end{aligned}$$

Equating coefficients:  $6 = 2P \Rightarrow \underline{P=3}$       $Q-P=1 \Rightarrow \underline{Q=4}$   
 $P+R=2 \Rightarrow \underline{R=-1}$       $\rightarrow$

$$3 + \frac{4x-1}{2x^2-x+1}$$

$$\textcircled{4} \textcircled{a} \textcircled{i} \quad (1+x)^{\frac{3}{2}} \approx 1 + \left(\frac{3}{2}\right)x + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^2}{2!}$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\textcircled{ii} \quad (16+9x)^{\frac{3}{2}} = \left(16\left(1 + \frac{9}{16}x\right)\right)^{\frac{3}{2}} = 64\left(1 + \frac{9}{16}x\right)^{\frac{3}{2}}$$

now replace  $x$  with  $\frac{9}{16}x$  in  $\textcircled{a} \textcircled{i}$  and  $\times 64$ .

$$64\left(1 + \frac{3}{2}\left(\frac{9}{16}x\right) + \frac{3}{8}\left(\frac{9}{16}x\right)^2\right)$$

$$= 64\left(1 + \frac{27}{32}x + \frac{243}{2048}x^2\right)$$

$$= 64 + 54x + \frac{243}{32}x^2$$

$$\textcircled{b} \quad \text{let } 16+9x = 3 \quad \Rightarrow \quad 9x = -3 \quad \rightarrow \quad x = \frac{-1}{3}$$

$$3^{\frac{3}{2}} \approx 64 + 54\left(\frac{-1}{3}\right) + \frac{243}{32}\left(\frac{-1}{3}\right)^2$$

$$= 64 - 18 + \frac{27}{32}$$

$$= 46 + \frac{27}{32}$$

$$\underline{a=27} \quad \underline{b=32}$$

$$\textcircled{5} \textcircled{a} \textcircled{i} \quad \text{use double angle formula: } \cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$3\cos 2x + 2\sin x + 1 = 0$$

$$3(1 - 2\sin^2 x) + 2\sin x + 1 = 0$$

$$3 - 6\sin^2 x + 2\sin x + 1 = 0 \quad \rightarrow \quad 6\sin^2 x - 2\sin x - 4 = 0$$

$$3\sin^2 x - \sin x - 2 = 0$$

$$(ii) \text{ let } x = \sin \alpha$$

$$3x^2 - x - 2 = 0$$

$$(3x+2)(x-1) = 0$$

$$x = \sin \alpha = \underline{\underline{1}} \quad \text{or} \quad \underline{\underline{-\frac{2}{3}}}$$

$$(b)(i) \quad R \cos(2\alpha - \alpha) = R(\cos(2\alpha)\cos \alpha + \sin(2\alpha)\sin \alpha) \\ = (R \cos \alpha) \cos 2\alpha + (R \sin \alpha) \sin 2\alpha$$

$$3 \cos 2\alpha + 2 \sin 2\alpha = (R \cos \alpha) \cos 2\alpha + (R \sin \alpha) \sin 2\alpha$$

$$\textcircled{1} \quad 3 = R \cos \alpha$$

$$\textcircled{2} \quad 2 = R \sin \alpha$$

$$\frac{\textcircled{2}}{\textcircled{1}} : \quad \frac{2}{3} = \frac{\sin \alpha}{\cos \alpha} \Rightarrow \tan \alpha = \frac{2}{3} \quad \underline{\underline{\alpha = 33.7^\circ}}$$

$$\textcircled{1}^2 + \textcircled{2}^2 : \quad 3^2 + 2^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha$$

$$13 = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$13 = R^2 \Rightarrow \underline{\underline{R = \sqrt{13}}}$$

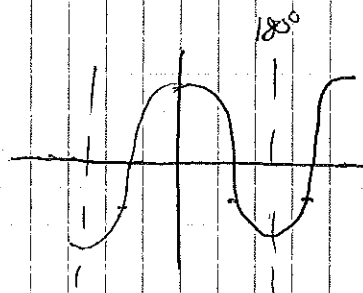
$$(ii) \quad \sqrt{13} \cos(2\alpha - 33.7^\circ) + 1 = 0$$

$$\cos(2\alpha - 33.7) = \frac{-1}{\sqrt{13}}$$

$$2\alpha - 33.7 = \pm 106.1^\circ$$

$$253.9^\circ$$

$$\alpha = 69.9^\circ, \quad \underline{\underline{143.8^\circ}}$$



$$\textcircled{6} \text{ a) } x^3 y + \cos(\pi y) = 7$$

$$\underline{y=1} : x^3 + \cos \pi = 7$$

$$x^3 = 7 - \cos \pi = 8$$

$$\Rightarrow \underline{\underline{x=2}}$$

$$\text{(b) } \frac{d}{dx} (x^3 y) = \left( \frac{d}{dx} x^3 \right) xy + x^3 \times \left( \frac{d}{dx} y \right)$$

← product rule.

$$= 3x^2 y + x^3 \left( \frac{dy}{dx} \frac{d}{dy} y \right)$$

$$= 3x^2 y + x^3 \frac{dy}{dx}$$

$$\frac{d}{dx} [\cos(\pi y)] = \frac{dy}{dx} \frac{d}{dy} (\cos(\pi y)) = -\pi \sin(\pi y) \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx} (x^3 y + \cos(\pi y)) = \frac{d}{dx} 7$$

$$3x^2 y + x^3 \frac{dy}{dx} - \pi \sin(\pi y) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} \times [x^3 - \pi \sin(\pi y)] = -3x^2 y$$

$$\frac{dy}{dx} = \frac{3x^2 y}{\pi \sin(\pi y) - x^3}$$

$$\underline{y=1} \rightarrow \underline{x=2} \quad (\text{from (a)})$$

$$\frac{dy}{dx} = \frac{3(4)}{\pi \sin \pi - 8} = \frac{12}{-8} = \underline{\underline{-\frac{3}{2}}}$$

7(a) B:  $\begin{pmatrix} -1+\mu \\ 3-2\mu \\ 4-\mu \end{pmatrix}$   $\mu=2 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

(b) (i)  $r_1 = \begin{pmatrix} 4+2\lambda \\ -3 \\ 2+\lambda \end{pmatrix}$   $r_2 = \begin{pmatrix} -1+\mu \\ 3-2\mu \\ 4-\mu \end{pmatrix}$

①  $4+2\lambda = -1+\mu$

②  $-3 = 3-2\mu$

③  $2+\lambda = 4-\mu$

Solve ① and ② simultaneously and then check in ③

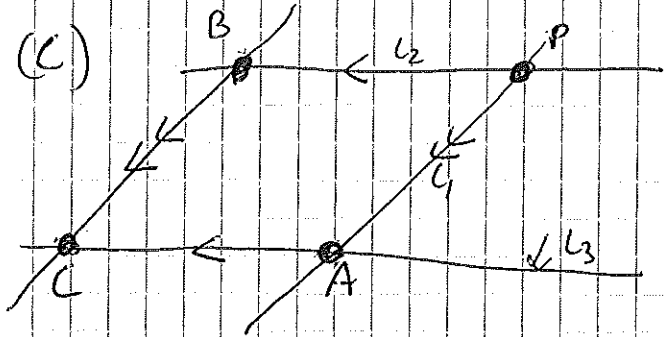
②:  $-6 = -2\mu \Rightarrow \underline{\mu=3}$

①:  $4+2\lambda = -1+3$   
 $2\lambda = -2 \Rightarrow \underline{\lambda = -1}$

Check ③:  $2+(-1) = 1$   
 $4-(3) = 1$

✓ They do intersect.

(ii) Intersect when  $\lambda = -1$ :  $\begin{pmatrix} 4-2 \\ -3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$   
 P is  $(2, -3, 1)$



$$r_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2t \\ -1 \\ 2+t \end{pmatrix}$$

$\uparrow$  B lies on line       $\uparrow$  parallel to  $L_1$

$$\vec{OC} = \begin{pmatrix} 1+2g \\ -1 \\ 2+g \end{pmatrix} \quad \text{for some value } g$$

$$|\vec{CB}| = |\vec{AP}|$$

$$\vec{AP} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow |\vec{AP}| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$$

$$\vec{CB} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1+2g \\ -1 \\ 2+g \end{pmatrix} = \begin{pmatrix} -2g \\ 0 \\ -g \end{pmatrix}$$

$$|\vec{CB}| = \sqrt{5} \Rightarrow \sqrt{(-2g)^2 + (-g)^2} = \sqrt{5g^2} = \sqrt{5}$$

$$5g^2 = 5 \Rightarrow g^2 = 1$$

$$g = \pm 1$$

$$C_1 = \begin{pmatrix} 1+2(1) \\ -1 \\ 2+(1) \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1+2(-1) \\ -1 \\ 2+(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$(8) (a) \frac{dx}{dt} = \frac{-1}{5} (x+1)^{\frac{1}{2}}$$

Separate variables

$$\int (x+1)^{-\frac{1}{2}} dx = \int \frac{-1}{5} dt$$

$$2(x+1)^{\frac{1}{2}} = \frac{-1}{5} t + C$$

$$(x+1)^{\frac{1}{2}} = \frac{-1}{10} t + k \Rightarrow x+1 = \left(\frac{-1}{10} t + k\right)^2$$

$$x = \left(\frac{-1}{10} t + k\right)^2 - 1$$

$$\underline{t=0} \quad \underline{x=80} \Rightarrow 80 = k^2 - 1$$

$$\Rightarrow k = 9$$

$$x = \left(9 - \frac{1}{10} t\right)^2 - 1$$

$$(b) \underline{t=60}$$

$$x = \left(9 - \frac{1}{10}(60)\right)^2 - 1 = 3^2 - 1 = 8\%$$

$$(c) (i) \frac{dA}{dt} \propto A(9-A) \Rightarrow \frac{dA}{dt} = K \cdot A(9-A) \quad \text{for constant } K$$

$$(ii) 4.5 = \frac{9}{1 + 4e^{-0.09t}} \Rightarrow 4.5 + 18e^{-0.09t} = 9$$

$$18e^{-0.09t} = 4.5$$

$$e^{-0.09t} = \frac{4.5}{18} = \frac{1}{4}$$

$$\ln(e^{-0.09t}) = \ln\left(\frac{1}{4}\right)$$

$$-0.09t = \ln\left(\frac{1}{4}\right)$$

$$t = \frac{-\ln\left(\frac{1}{4}\right)}{0.09} = \frac{\ln 4}{0.09} = \frac{100}{9} \ln 4 = 15.4 \text{ hours.}$$