Core 4 - June 2009

① a) $3x - 1 \rightarrow$ sub in $x = \frac{1}{3}$

$$f(1/3) = 3(1/3)^3 + 8(1/3)^2 - 3(1/3) - 5$$
$$= -5$$

b) Long Division:

$$\begin{array}{r} x^2 + 3x \\ (3x-1) \overline{\smash{)}\, 3x^3 + 8x^2 - 3x - 5} \\ 3x^3 - x^2 \\ + 9x^2 - 3x \end{array}$$

$$\Rightarrow x^2 + 3x - \frac{5}{3x-1}$$

② a) $x = 1/t$ $\qquad$ $y = t + 1/2t$

$dx/dt = -1/t^2$ $\qquad$ $dy/dt = 1 - \frac{1}{2t^2}$

$$dy/dx = dy/dt \times dt/dx$$
$$= \left(1 - \frac{1}{2t^2}\right) \times -t^2$$
$$= -t^2 + 1/2 \quad \text{or} \quad 1/2 - t^2$$

b) $t = 1$, $dy/dx = 1/2 - 1 = -1/2$

$\therefore$ gradient of normal $= 2$

$x = 1/1 = 1$

$y = 1 + 1/2 = 3/2$

$m = 2$

$y - 3/2 = 2(x - 1)$

$y - 3/2 = 2x - 2$

$y = 2x - 1/2$

c) $x = 1/t \rightarrow t = 1/x$

$y = t + 1/2t$

$y = 1/x + \frac{1}{2(1/x)}$

$y = 1/x + \frac{x}{2}$

$xy = 1 + x^2/2$

$2xy = 2 + x^2$

$\rightarrow x^2 - 2xy + 2 = 0$

3) a) $(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2$

$= 1 + x + x^2$

b) i) $\frac{3x-1}{(1-x)(2-3x)} = \frac{A}{1-x} + \frac{B}{2-3x}$

$3x - 1 = A(2-3x) + B(1-x)$

$x = 1$ $\quad 2 = A(2-3) \rightarrow A = -2$

$x = 2/3$ $\quad 3(2/3) - 1 = B(1 - 2/3)$

$1 = B(1/3) \rightarrow B = 3$

$= \frac{-2}{1-x} + \frac{3}{2-3x}$

ii) $\frac{-2}{1-x} + \frac{3}{2-3x}$

$= -2(1-x)^{-1} + 3(2-3x)^{-1}$

$3(2)^{-1}\left[1 - \frac{3}{2}x\right]^{-1}$

$-2\left[1 + x + x^2\right]$

$= -2 - 2x - 2x^2$ $\qquad = \frac{3}{2}\left[1 + \frac{3}{2}x + \frac{9}{4}x^2\right]$

$= \frac{3}{2} + \frac{9}{4}x + \frac{27}{8}x^2$

$\boxed{\text{TOTAL}}$ $-\frac{1}{2} + \frac{1}{4}x + \frac{11}{8}x^2$

c) $-1 < x < 1$ AND $-1 < \frac{3}{2}x < 1$

$-2 < 3x < 2$

$-\frac{2}{3} < x < \frac{2}{3}$

Always choose most restricted

$\rightarrow -\frac{2}{3} < x < \frac{2}{3}$ or $|x| < \frac{2}{3}$

(4) a) i) $V = Ak^t$, $V = 12499$ when $t = 0$

$\rightarrow 12499 = A$

ii) $V = 12499k^t$, $V = 7000$ when $t = 36$

$\rightarrow 7000 = 12499K^{36}$

$\frac{7000}{12499} = K^{36}$

$K = \sqrt[36]{\frac{7000}{12499}} = 0.984025$ (6dp)

b) $12499 \times 0.984025^n < 5000$

$0.984025^n < \frac{5000}{12499}$

$n \ln(0.984025) < \ln\left(\frac{5000}{12499}\right)$

negative so swap!

$n > \ln\left(\frac{5000}{12499}\right) \div \ln(0.984025)$

$n > 56.893$, $\therefore n = 57$

(5) $4x^2 + y^2 = 4 + 3xy$

$8x + 2y\,{}^{dy}/_{dx} = 3x\,{}^{dy}/_{dx} + 3y$

$u = 3x$ $v = y$

${}^{du}/_{dx} = 3$ ${}^{dv}/_{dx} = {}^{dy}/_{dx}$

${}^{dy}/_{dx} = 3x\,{}^{dy}/_{dx} + 3y$

when $x = 1$ and $y = 3$

$\rightarrow 8 + 6\,{}^{dy}/_{dx} = 3\,{}^{dy}/_{dx} + 9$

$3\,{}^{dy}/_{dx} = 1$

${}^{dy}/_{dx} = 1/3$

(6) a) i) $3\cos(2x) + 7\cos(x) + 5 = 0$

$3(2\cos^2(x) - 1) + 7\cos(x) + 5 = 0$

$6\cos^2(x) - 3 + 7\cos(x) + 5 = 0$

$6\cos^2(x) + 7\cos(x) + 2 = 0$

ii) $(3\cos(x) + 2)(2\cos(x) + 1) = 0$

$\cos(x) = -2/3$ $\cos(x) = -1/2$

b) i) $7\sin(\theta) + 3\cos(\theta) = R\sin(\theta + \alpha)$

$= R\left(\sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha)\right)$

$\therefore\ 7 = R\cos(\alpha)$

$3 = R\sin(\alpha)$

$\Rightarrow \tan(\alpha) = 3/7 \rightarrow \alpha = 23.2^\circ$ (1dp)

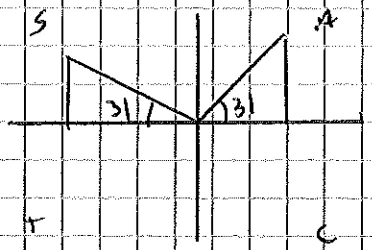
$R = \sqrt{3^2 + 7^2} = \sqrt{58}$

$\Rightarrow \sqrt{58}\sin(\theta + 23.2)$

ii) $\sqrt{58}\sin(\theta + 23.2) = 4$

$\sin(\theta + 23.2) = 4/\sqrt{58}$

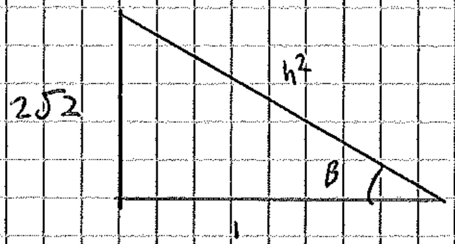
Key angle = 31.683



Angles = 31.683 , 148.316

$\therefore\ \theta = 8.5^\circ ,\ 125.1^\circ$ (1dp)

c) i)



$\tan\beta = \frac{OPP}{ADJ} = \frac{2\sqrt{2}}{1}$

$h^2 = \sqrt{(2\sqrt{2})^2 + 1^2}$

$h^2 = \sqrt{9} = 3$

$\therefore\ \cos\beta = \frac{ADJ}{HYP} = \frac{1}{3}$

ii) $\sin 2\beta = 2\sin\beta\cos\beta$

$\sin\beta = \frac{OPP}{HYP} = \frac{2\sqrt{2}}{3}$

$= 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right)$

$= \frac{4}{9}\sqrt{2}$

7 a) $\vec{AB} = \vec{AO} + \vec{OB}$

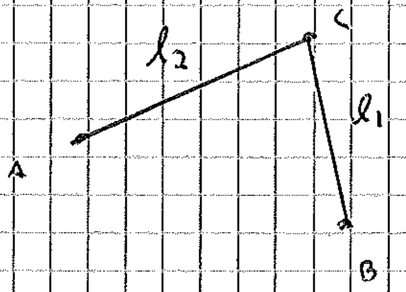
$$= \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$|\vec{AB}| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$

b) $$\begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{array}{l} 6 + 2\lambda = 4 \\ -1 - \lambda = 0 \\ 5 + 4\lambda = 1 \end{array}\right\}$$ All satisfied by $\lambda = -1$

$6 - 2 = 4$ ✓
$-1 + 1 = 0$ ✓
$5 - 4 = 1$ ✓

c) 

Find C:

$$\begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ -8 \end{pmatrix}$$

$= 6 + 2\lambda = 3 - \mu$ (1)

$-1 - \lambda = -2 + 3\mu$ (2)

$5 + 4\lambda = 5 - 8\mu$ (3)

use (1) & (2)

(1) $6 + 2\lambda = 3 - \mu$

$2\times$(2) $-2 - 2\lambda = -4 + 6\mu$ +

$4 = -1 + 5\mu \rightarrow 5\mu = 5 \rightarrow \mu = 1$

use (1) to find λ

$6 + 2\lambda = 3 - 1$

$2\lambda = -4 \rightarrow \lambda = -2$

Check in (3): $5 + 4(-2) = 5 - 8(1)$

$-3 = -3$ ✓

So, C must be: $$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 3 \\ -8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = (2, 1, -3)$$

$\vec{BC} = \vec{BO} + \vec{OC}$

$= \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

$|\vec{BC}| = \sqrt{(-2)^2 + 1^2 + (-4)^2} = \sqrt{21}$

$|\vec{BC}| = |\vec{AB}|$ $\therefore$ Triangle is Isosceles.

(8) a) $\frac{dx}{dt} = \frac{150 \cos(2t)}{x}$

$\Rightarrow \int x \, dx = \int 150 \cos(2t) \, dt$

$\Rightarrow x^2/2 = 75 \sin(2t) + c$

$x = 20, \quad t = \pi/4$

$400/2 = 75 \sin(\pi/2) + c$

$200 = 75 + c \Rightarrow c = 125$

$\therefore x^2/2 = 75 \sin(2t) + 125$

$x^2 = 150 \sin(2t) + 250$

b) i) $t = 13 \Rightarrow x^2 = 150 \sin(26) + 250$

$x^2 = 364.38\ldots \Rightarrow x = 19.1$ cm (1dp)

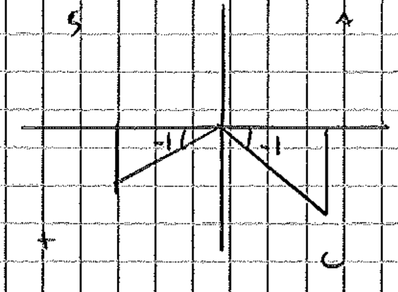
ii) $x = 11 \Rightarrow 11^2 = 150 \sin(2t) + 250$

$-129 = 150 \sin(2t)$

$\sin(2t) = -\frac{129}{150}$

$2t = \sin^{-1}\left(\frac{-129}{150}\right)$

$2t = -1.035$



First positive : $4.17686\ldots = 2t$

$\therefore t = 2.088 = 2.1$ sec (1dp)