Core 4 - January 2009

① a) i) $f(-1) = 4(-1)^3 - 7(-1) - 3 = 0$

ii) $2x+1 \rightarrow$ sub in $x = -1/2$

$f(-1/2) = 4(-1/2)^3 - 7(-1/2) - 3$ $\therefore$ FACTOR

$\rightarrow 4(-1/8) + 7/2 - 3 = 0$

iii) Need 3rd bracket

$(x+1)(2x+1)(\quad) = 4x^3 + 0x^2 - 7x - 3$

$(2x^2+3x+1)(\quad) = 4x^3 + 0x^2 - 7x - 3$

$2x$ $\quad$ -3

$\rightarrow \dfrac{(x+1)(2x+1)(2x-3)}{(x+1)(2x+1)} = 2x - 3$

b) $2x+1 \rightarrow$ sub in $x = -1/2$

using remainder theorem:

$g(-1/2) = 4(-1/2)^3 - 7(-1/2) + d = 2$

$-1/2 + 7/2 + d = 2$

$\rightarrow d = -1$

② a) $\sin(x) - 3\cos(x) = R\sin(x-\alpha)$

$= R\sin(x)\cos(\alpha) - R\cos(x)\sin(\alpha)$

$R = \sqrt{1^2 + 3^2} = \sqrt{10}$

$1 = R\cos(\alpha)$
$3 = R\sin(\alpha)$ $\rightarrow \tan(\alpha) = 3/1 = 3$

$\rightarrow \alpha = 1.249$ (rad)

$\rightarrow \sqrt{10}\sin(x - 1.249)$

b) i) Minimum $= -\sqrt{10}$ as curve stretched SF $\sqrt{10}$ in y

ii) $y = \sin(x)$ has minimum value at $x = 3\pi/2$

Function has been translated 1.249 to right

$\therefore$ new minimum $= 3\pi/2 + 1.249 = 5.961\ldots$

$= 5.96$ (2dp)

(3) a) i) $\frac{2x+7}{x+2} = A + \frac{B}{x+2}$

$\rightarrow \quad 2x+7 = A(x+2) + B$

$\boxed{x=-2}$ $\quad 3 = B$

$\boxed{x=0}$ $\quad 7 = 2A + 3 \rightarrow A = 2$

$\rightarrow \quad 2 + \frac{3}{x+2}$

ii) $\int 2 + \int \frac{3}{x+2} = 2x + 3\ln(x+2) + C$

b) i) $\frac{28+4x^2}{(1+3x)(5-x)^2} = \frac{P}{1+3x} + \frac{Q}{5-x} + \frac{R}{(5-x)^2}$

$28 + 4x^2 = P(5-x)^2 + Q(1+3x)(5-x) + R(1+3x)$

$\boxed{x=5}$ $\quad 128 = R(16) \rightarrow R = 8$

$\boxed{x=-1/3}$ $\quad 256/9 = P(256/9) \rightarrow P = 1$

$\boxed{x=0}$ $\quad 28 = (1)(5)^2 + Q(1)(5) + 8(1)$

$\rightarrow 28 = 25 + 5Q + 8$

$\rightarrow 5Q = -5 \rightarrow Q = -1$

$\rightarrow \frac{1}{1+3x} - \frac{1}{5-x} + \frac{8}{(5-x)^2}$

ii) $\int \frac{1}{1+3x} - \int \frac{1}{5-x} + 8\int \frac{1}{(5-x)^2} \rightsquigarrow 8\int (5-x)^{-2}$

$\rightarrow \frac{1}{3}\ln(1+3x) + (-\ln(5-x)) - 8(5-x)^{-1} + C$

$\rightarrow \frac{1}{3}\ln(1+3x) + \ln(5-x) - \frac{8}{5-x} + C$

(4) a) i) $(1-x)^{1/2} \approx 1 + (1/2)(-x) + \frac{(1/2)(-1/2)}{2!}(-x)^2$

$= 1 - 1/2\,x - 1/8\,x^2$

ii) $\sqrt{4-x} = (4-x)^{1/2} = 4^{1/2}(1 - x/4)^{1/2}$

$= 2\left[1 - 1/2(x/4) - 1/8(x/4)^2\right]$

$= 2\left[1 - x/8 - x^2/128\right]$

$= 2 - x/4 - x^2/64$

b) To make $\sqrt{4-x}$ approximate $\sqrt{3}$, $x = 1$

$\Rightarrow 2 - \frac{1}{4} - \frac{1}{64}$

$= 1.734375 = 1.734$ (3dp)

(5) a) $\sin(2x) = 2\sin(x)\cos(x)$

b) $5\sin(2x) + 3\cos(x) = 0$

$5(2\sin(x)\cos(x)) + 3\cos(x) = 0$

$\rightarrow 10\sin(x)\cos(x) + 3\cos(x) = 0$

$\cos(x)\left[10\sin(x) + 3\right] = 0$

$\cos(x) = 0$

$\rightarrow x = 90°, 270°$

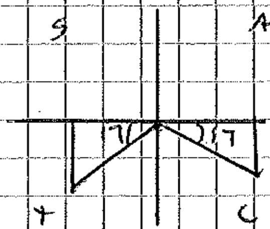
$10\sin(x) + 3 = 0$

$\rightarrow \sin(x) = -3/10$

$x = -17.45$

$x = 342.54, \quad x = 197.45$

$= 342.5°, \quad 197.5°$ (1dp)



c) $\sin(2x) + \cos(2x) = 1 + \sin(x)$

$2\sin(x)\cos(x) + (1 - 2\sin^2(x)) = 1 + \sin(x)$

$2\sin(x)\cos(x) - 2\sin^2(x) = \sin(x)$

$2\sin(x)\left[\cos(x) - \sin(x)\right] = \sin(x)$

$2\left[\cos(x) - \sin(x)\right] = 1$ $\{\div$ by $\sin(x)\}$

(6) a) $x^2y + y^3 = 2x + 1$

$\rightarrow 2xy + x^2y\, dy/dx + 3y^2\, dy/dx = 2$

when $x = 2$, $y = 1$

$\rightarrow 4 + 4\, dy/dx + 3\, dy/dx = 2$

$7\, dy/dx = -2$

$dy/dx = -2/7$

x^2y

$u = x^2 \quad v = y$

$du/dx = 2x \quad dv/dx = y\frac{dy}{dx}$

$\rightarrow 2xy + x^2y\, dy/dx$

b) At st point, $dy/dx = 0$

$\rightarrow 2xy = 2$ (from equation in a)

$\rightarrow y = 1/x$

Sub in to equation of curve:

$\rightarrow x^2 (1/x) + (1/x)^3 = 2x + 1$

$\rightarrow x + 1/x^3 = 2x + 1$

$\rightarrow 1/x^3 = x + 1$

(7) i) $dx/dt = -kt\, e^{1/2 x}$

$\frac{1}{e^{1/2 x}}\, dx = -kt\, dt$

$\int e^{-1/2 x}\, dx = -k \int t\, dt$

$= -2e^{-1/2 x} = -k\, t^2/2 + c$

ii) $x = 6,\ t = 0 \rightarrow -2e^{-3} = 0 + c \rightarrow c = -2e^{-3}$

$-2e^{-1/2 x} = -k\, t^2/2 - 2e^{-3}$

÷ by −2 $\cancel{\ln(-2e^{-1/2x}) = \ln(k\, t^2/2 - 2e^{-3})}$

$\rightarrow e^{-1/2 x} = k\, t^2/4 + e^{-3}$

$\ln(e^{-1/2 x}) = \ln(k\, t^2/4 + e^{-3})$

$\rightarrow -1/2\, x = \ln(k\, t^2/4 + e^{-3})$

$\rightarrow x = -2 \ln(k\, t^2/4 + e^{-3})$

b) i) $x = -2 \ln\left(\frac{0.004 \times 10^2}{4} + e^{-3}\right)$

$= 3.797\ldots = 3797 = 3800$ (nearest 100)

ii) Need $x = 0$

$\rightarrow 0 = -2 \ln\left(\frac{0.004 \times t^2}{4} + e^{-3}\right)$

$0 = \ln\left(\frac{0.004 \times t^2}{4} + e^{-3}\right)$

=1 $\rightarrow \ln(0) = \frac{0.004 \times t^2}{4} + e^{-3}$

$$1 - e^{-3} = \frac{0.004\,t^2}{4}$$

$$4(1 - e^{-3}) = 0.004\,t^2$$

$$\Rightarrow t = \sqrt{\frac{4(1-e^{-3})}{0.004}}$$

$$= 30.8255\ldots = 30.8 \ (1dp)$$

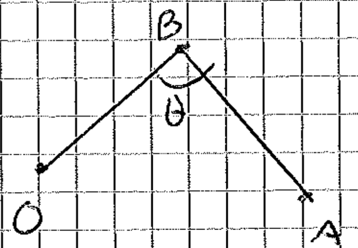
(8) a) i) $\vec{AB} = \vec{AO} + \vec{OB}$

$$= \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

ii) $\cancel{|a| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}}$

$$|b| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$$

$$|AB| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$



Need scalar product of $\vec{AB}$ & $\vec{OB}$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 3 + 0 + 2 = 5$$

$$a \cdot b = |a|\,|b| \cos(\theta)$$

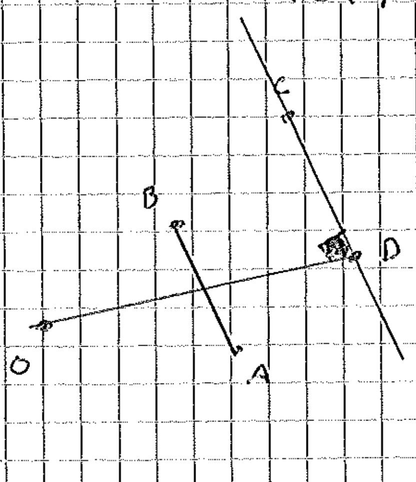
$$\Rightarrow 5 = \sqrt{14} \times \sqrt{2} \times \cos(\theta)$$

$$\Rightarrow \cos(\theta) = \frac{5}{\sqrt{14}\sqrt{2}} = \frac{5}{\sqrt{28}} = \frac{5}{2\sqrt{7}}$$

b) $\vec{OC} = 2\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$ Parallel to AB

$$\therefore l = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

c)



As D is on l; $\vec{OD} = \begin{pmatrix} 6 + \lambda \\ 2 \\ -4 - \lambda \end{pmatrix}$

As $\vec{DC}$ is parallel to $\vec{AB}$ and ODC is 90°:

$$\vec{OD} \cdot \vec{AB} = 0$$

$$\begin{pmatrix} 6 + \lambda \\ 2 \\ -4 - \lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 6 + \lambda + 4 + \lambda = 0$$

$\rightarrow \quad 10 + 2\lambda = 0$

$\rightarrow \lambda = -5$

$$\therefore \quad \vec{OD} = \begin{pmatrix} 6 - 5 \\ 2 \\ -4 - (-5) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ co-ordinates of D are $(1, 2, 1)$