Core 4 - June 2008

① a) $3x+1 \Rightarrow$ sub in $x = -1/3$

$$f(-1/3) = 27(-1/3)^3 - 9(-1/3) + 2 = 4$$

b) i) $f(-2/3) = 27(-2/3)^3 - 9(-2/3) + 2 = 0$

ii) If $-2/3$ has no remainder, $(3x+2)$ must be a factor

$$(3x+2)(ax^2+b+c) = 27x^3 - 9x + 2$$

Division:

$$\begin{array}{r|llll} & 9x^2 & -6x & +1 & \\ 3x+2 & 27x^3 & +0x^2 & -9x & +2 \\ & 27x^3 & +18x^2 & & \\ & & -18x^2 & -12x & \\ & & & +3x & +2 \end{array}$$

$$f(x) = (3x+2)(9x^2-6x+1)$$

$$= (3x+2)(3x-1)(3x-1)$$

iii) $$\frac{(3x+2)(3x-1)(3x-1)}{(3x-1)(3x+2)} = 3x-1$$

② a) $x = 4t+3$ $\qquad$ $y = 1/2t - 1 = 1/2\, t^{-1} - 1$

$dx/dt = 4$ $\qquad$ $dy/dt = -1/2\, t^{-2}$

$$= -1/2t^2$$

$$dy/dx = dy/dt \times dt/dx$$

$$= -1/2t^2 \times 1/4 = -1/8t^2$$

when $t = 1/2$, $dy/dx = -1/8(1/2)^2 = -1/2$

b) At P, $t = 1/2$

$x = 4(1/2) + 3 = 5$

$y = 1/2(1/2) - 1 = 0$

$dy/dx = -1/2$, $\therefore$ gradient of normal $= 2$

$$y - y_1 = m(x - x_1)$$

$$y = 2(x-5) \Rightarrow y = 2x - 10$$

c) $y = \dfrac{1}{2t} - 1$ $\qquad x = 4t + 3$

$\quad$ (2) $4t = x - 3$

$\rightarrow y + 1 = 1/2t$

(1) $y + 1 = 2/4t$

sub (2) into (1) $\Rightarrow y + 1 = 2/(x-3)$

$(y+1)(x-3) = 2$

(3) a) $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$\sin(x + 2x) = \sin(x)\cos(2x) + \cos(x)\sin(2x)$

$\cos(2x) = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\Rightarrow \quad = \sin(x)(1 - 2\sin^2(x)) + \cos(x)(2\sin(x)\cos(x))$

$= \sin(x) - 2\sin^3(x) + 2\sin(x)\cos^2(x)$

$\cos^2(x) = 1 - \sin^2(x)$

$= \sin(x) - 2\sin^3(x) + 2\sin(x)(1 - \sin^2(x))$

$= \sin(x) - 2\sin^3(x) + 2\sin(x) - 2\sin^3(x)$

$= 3\sin(x) - 4\sin^3(x)$

b) $\int \sin^3(x)\,dx$

$\sin(3x) = 3\sin(x) - 4\sin^3(x)$

$4\sin^3(x) = 3\sin(x) - \sin(3x)$

$\sin^3(x) = \cancel{3/4}\ 1/4\left[3\sin(x) - \sin(3x)\right]$

$= 1/4 \int 3\sin(x) - \sin(3x)\,dx$

$= 1/4\left[-3\cos(x) + 1/3\cos(3x)\right]$

$= -3/4\cos(x) + 1/12\cos(3x) + c$

(4) a) i) $(1 - x)^{1/4} \approx 1 + (1/4)(-x) + \dfrac{(1/4)(-3/4)}{2!}(-x)^2$

$= 1 - x/4 - 3/32\, x^2$

ii) $(81 - 16x)^{1/4} = 81^{1/4}\left(1 - 16/81\, x\right)^{1/4}$

$= 3\left[1 - 1/4\,(16/81\, x) - 3/32\,(16/81\, x)^2\right]$

$= 3\left[1 - 4/81\, x - 8/2187\, x^2\right]$

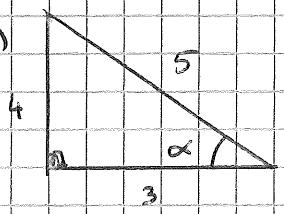
$= 3 - 4/27\, x - 8/729\, x^2$

b) $(81-16x)^{1/4}$ let $x = 1/16$ $\rightarrow$ $(81-1)^{1/4} = \sqrt[4]{80}$

$x = 1/16 \rightarrow 3 - 4/27\,(1/16) - 8/729\,(1/16)^2$

$= 2.99066979$

(5) a) i)



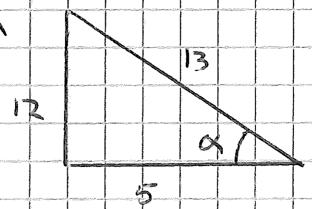
$\sin\alpha = 4/5$

$\Rightarrow \cos\alpha = 3/5$

ii) $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$= 3/5 \cos(\beta) + 4/5 \sin(\beta)$

iii)



$\cos\beta = 5/13$

$\sin\beta = 12/13$

$\rightarrow \cos(\alpha - \beta) = 3/5 \times 5/13 + 4/5 \times 12/13$

$= 63/65$

b) i) $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)} = 1$

$\rightarrow 2\tan(x) = 1 - \tan^2(x)$

$\rightarrow \tan^2(x) + 2\tan(x) - 1 = 0$

ii) $\tan(2x) = 1 = \tan(45)$

$\therefore x = 22.5$

$\tan^2(x) + 2\tan(x) - 1 = 0$

Doesn't factorise, so use formula:

$\tan(x) = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-1)}}{2}$

$\tan(x) = \dfrac{-2 \pm \sqrt{8}}{2} = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

As we know $x = 22.5°$, and hence x is acute, the answer must be the positive root

so $\tan(x) = -1 + \sqrt{2}$ or $\sqrt{2} - 1$

(6) a) $\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+1)}$

$2 \equiv A(x+1) + B(x-1)$

$\boxed{x=-1}$ $\quad 2 \equiv B(-2) \rightarrow B = -1$

$\boxed{x=1}$ $\quad 2 \equiv A(2) \rightarrow A = 1$

$\rightarrow \frac{1}{x-1} - \frac{1}{x+1}$

b) $\int \frac{1}{x-1} - \frac{1}{x+1} \, dx$

$= \ln(x-1) - \ln(x+1)$

c) $\frac{dy}{dx} = \frac{2y}{3(x^2-1)}$

$\rightarrow \int \frac{3}{y} \, dy = \int \frac{2}{(x^2-1)} \, dx$

$\rightarrow 3\ln(y) = \ln(x-1) - \ln(x+1) + c$

when $x=3$, $y=1$

$\rightarrow 3\ln(1) = \ln(2) - \ln(4) + c$

$0 \quad \cancel{0 = \ln(2/4) + c}$

$\ln(4) - \ln(2) = c$

$c = \ln(4/2) = \ln(2)$

$\rightarrow 3\ln(y) = \ln(x-1) - \ln(x+1) + \ln(2)$

$\rightarrow \ln(y^3) = \ln\left(\frac{2(x-1)}{x+1}\right)$

$\rightarrow y^3 = \frac{2(x-1)}{x+1}$

(7) a) Distance $= \sqrt{(5-3)^2 + (3--2)^2 + (0-1)^2}$

$= \sqrt{4 + 25 + 1} = \sqrt{30}$

b) $\vec{AB} = \vec{AO} + \vec{OB} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

$|AB| = \sqrt{30}$

$|l| = \sqrt{1^2 + 0^2 + (-3)^2} = \sqrt{10}$

$$AB \cdot l = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = 2 + 0 + 3 = 5$$

$$\cos(\theta) = \frac{AB \cdot l}{|AB|\,|l|}$$

$$\cos(\theta) = \frac{5}{\sqrt{30}\,\sqrt{10}} = \frac{5}{\sqrt{300}}$$

$$\rightarrow \theta = \cos^{-1}\left(\frac{5}{\sqrt{300}}\right) = 73.221 = 73^\circ \text{ (nearest degree)}$$

c) $\vec{AC} = \vec{AO} + \vec{OC}$

$$\vec{OC} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 5+\lambda \\ 3 \\ -3\lambda \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 5+\lambda \\ 3 \\ -3\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2+\lambda \\ 5 \\ -1-3\lambda \end{pmatrix}$$

We know $|AC| = |AB| = \sqrt{30}$

$$\therefore (2+\lambda)^2 + 5^2 + (-1-3\lambda)^2 = 30$$

$$4 + 4\lambda + \lambda^2 + 25 + 1 + 6\lambda + 9\lambda^2 = 30$$

$$10\lambda^2 + 10\lambda + 30 = 30$$

$$\lambda^2 + \lambda = 0$$

$$\rightarrow \lambda(\lambda + 1) = 0$$

$\lambda = 0$ $\qquad\qquad$ $\lambda = -1$

$$\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = B$$

$$\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + -1\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} = C$$

(8) a) i) $\frac{\partial x}{\partial t} \propto -x$

$\rightarrow \frac{\partial x}{\partial t} = -Kx$

ii) $-500 = -K(20{,}000)$

$\rightarrow K = {}^{500}/_{20{,}000} = 0.025$

b) $P = 2000 - Ae^{-0.05t}$

i) $t = 0,\ P = 700$

$\rightarrow 700 = 2000 - Ae^{0}$

$A = 2000 - 700 = 1300$

ii) Need: $2000 - 1300e^{-0.05t} > 1900$

$\therefore 100 > 1300e^{-0.05t}$

$1/13 > e^{-0.05t}$

$\ln(1/13) > -0.05t$

< sign swaps as × by −20

$\rightarrow -20\ln(1/3) < t$

$\rightarrow t > 51.298..$

$\therefore$ population exceeds 1900 in $2008 + 51$

$= 2059$