Core 4 – June 2007

① a) $2x+1 \rightarrow$ sub in $x = -1/2$

$f(-1/2) = 2(-1/2)^2 + (-1/2) - 3 = -3$

b) $\dfrac{(2x+3)(x-1)}{(x+1)(x-1)} = \dfrac{2x+3}{x+1}$

② a) i) $(1+x)^{-1} = 1 + (-1)x + \dfrac{(-1)(-2)}{2!}(x)^2 + \dfrac{(-1)(-2)(-3)}{3!}(x)^3$

$= 1 - x + x^2 - x^3$

ii) $\dfrac{1}{1+3x} = (1+3x)^{-1} = 1 - (3x) + (3x)^2 - (3x)^3$

$= 1 - 3x + 9x^2 - 27x^3$

b) $\dfrac{1+4x}{(1+x)(1+3x)} = \dfrac{A}{1+x} + \dfrac{B}{1+3x}$

$1 + 4x = A(1+3x) + B(1+x)$

$\boxed{x = -1/3}$ $\quad -1/3 = B(2/3) \rightarrow B = -1/2$

$\boxed{x = -1}$ $\quad -3 = A(-2) \rightarrow A = 3/2$

$= \dfrac{3}{2(1+x)} - \dfrac{1}{2(1+3x)}$

c) i) $\dfrac{3}{2(1+x)} = 3/2\,(1+x)^{-1}$

$= 3/2\,[1 - x + x^2 - x^3]$

$= 3/2 - 3/2\,x + 3/2\,x^2 - 3/2\,x^3$

$\dfrac{-1}{2(1+3x)} = -1/2\,(1+3x)^{-1}$

$= -1/2\,[1 - 3x + 9x^2 - 27x^3]$

$= -1/2 + 3/2\,x - 9/2\,x^3 + 27/2\,x^3$

combined: $1 - 3x^2 + 12x^3$

ii) $|x| < 1$ and $|3x| < 1$

so take most limiting $\rightarrow |3x| < 1 \rightarrow |x| < 1/3$

(3) a) $4\cos(x) + 3\sin(x) = R\cos(x+\alpha)$

$= R[\cos(x)\cos(\alpha) + \sin(x)\sin(\alpha)]$

$R = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$4 = R\cos(\alpha)$
$3 = R\sin(\alpha)$ $\rightarrow$ $3/4 = \tan(\alpha)$

$\alpha = 36.869... = 36.9^\circ$

$\rightarrow 5\cos(x - 36.9^\circ)$

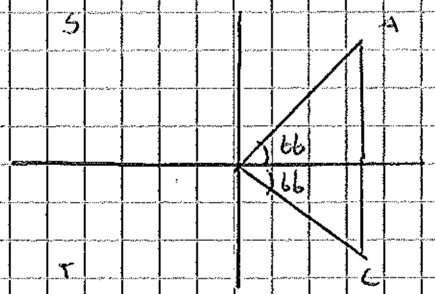
b) $5\cos(x - 36.9) = 2$

$\rightarrow \cos(x - 36.9) = 2/5$

$= \cos(t) = 0.4$ where $t = x - 36.9$

$t = 66.42..$ $\qquad 0 < x < 360$

$-36.9 < t < 323.1$



$t = 66.42, \; 293.57..$

$\boxed{x = t + 36.9}$

$= 103.3^\circ, \; 330.4^\circ$ (10p)

c) $5\cos(x-36.9)$ is stretch in y direction SF 5

$\rightarrow$ minimum value at -5

$\cos(x)$ has minimum value at $\cos^{-1}(-1) = 180^\circ$

$\therefore$ minimum value occurs at $180 + 36.9 = 216.9^\circ$

(4) a) i) $t = 0 \rightarrow x = 15 - 12e^0 = 15 - 12 = 3$ cm

ii) $t = 14 \rightarrow x = 15 - 12e^{-14/14} = 15 - 12e^{-1} = 10.6$ cm (3p)

b) i) $10 = 15 - 12e^{-t/14}$

$-5 = -12e^{-t/14}$

$5/12 = e^{-t/14}$

$\ln(5/12) = -t/14$

$14\ln(5/12) = -t$

$t = -14\ln(5/12) = 14\ln(5/12)^{-1} = 14\ln(12/5)$

ii) $t = 12.256... = 12$ days

c) i) $x = 15 - 12e^{-t/14}$

$dx/dt = -1/14 \times -12e^{-t/14}$

$\boxed{\begin{aligned} &\text{if } x = 15 - 12e^{-t/14} \\ &\Rightarrow x - 15 = -12e^{-t/14} \end{aligned}}$

$= -1/14\,(x - 15)$

$= 1/14\,(15 - x)$

ii) $x = 8 \Rightarrow 1/14\,(15-8) = 7/14 = 0.5$ cm per day

(5) a) $x = 1 \Rightarrow y + 4 = 5y^2$

$5y^2 - y - 4 = 0$

$(5y + 4)(y - 1) = 0$

$y = -4/5 \qquad y = 1 \rightarrow a = 1$ as $a > 0$

b) $y + 4x = 5x^2y^2$

$dy/dx + 4 = 10xy^2 + 10x^2y\, dy/dx$

PRODUCT RULE:

$u = 5x^2 \qquad v = y^2$

$du/dx = 10x \qquad dv/dx = 2y\, dy/dx$

$\rightarrow 10xy^2 + 10x^2y\, dy/dx$

$x = 1,\ y = 1$

$\rightarrow dy/dx + 4 = 10 + 10\, dy/dx$

$dy/dx - 10\, dy/dx = 10 - 4$

$dy/dx\,(1 - 10) = 6$

$dy/dx = 6/-9 = -2/3$

c) $dy/dx = -2/3$

$x = 1,\ y = 1 \rightarrow$

$y - y_1 = m(x - x_1)$

$y - 1 = -2/3\,(x - 1)$

$y - 1 = -2/3\,x + 2/3$

$y = -2/3\,x + 5/3$

(6) a) i) $x = \cos(\theta) \qquad y = \sin(2\theta)$

$dx/d\theta = -\sin(\theta) \qquad dy/d\theta = 2\cos(2\theta)$

ii) $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = 2\cos(2\theta) \times \dfrac{1}{-\sin(\theta)}$

when $\theta = \pi/6 \quad = 1 \times 1/-1/2 = -2$

b) $y = \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

- $x = \cos(\theta)$
- $\sin^2(\theta) + \cos^2(\theta) = 1 \rightarrow \sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1-x^2}$

$\therefore y = 2\sqrt{1-x^2} \times x$

$y^2 = 4(1-x^2)x^2$

$= 4x^2(1-x^2) \qquad k = 4$

(7) a) use scalar product: $\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 3 - 6 + 3 = 0$

$a.b = 0 \quad \therefore$ perpendicular

b) at intersection: ① $8 + 3\lambda = -4 + \mu$

② $6 - 3\lambda = 2\mu$

③ $-9 - \lambda = 11 - 3\mu$

use ① + ② ④
$$\begin{aligned} 8 + 3\lambda &= -4 + \mu \\ 6 - 3\lambda &= 2\mu \\ \hline 14 &= -4 + 3\mu \\ 18 &= 3\mu \rightarrow \mu = 6 \end{aligned}$$

use ② $6 - 3\lambda = 12$

$-6 = 3\lambda \rightarrow \lambda = -2$

check in ③ $-9 - \lambda = 11 - 3\mu$

$-9 + 2 = 11 - 18$ ✓

$\therefore$ Intersection: $8 + 3\lambda = 8 + 3(-2) = 2$

$6 - 3\lambda = 6 - 3(-2) = 12$

$-9 - \lambda = -9 - (-2) = -7$

$= (2, 12, -7)$

(*) c) ~~$\vec{AP} = \vec{AO} + \vec{OP} = \begin{pmatrix} 4 \\ 0 \\ -11 \end{pmatrix} + \begin{pmatrix} 2 \\ 12 \\ -7 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ -18 \end{pmatrix}$~~

~~$\vec{AB} = \vec{AP} + \vec{PB}$~~ ~~$|AP| = \sqrt{6^2 + 12^2 + (-18)^2}$~~

~~$\therefore |AB| = |AP| + |PB|$~~ ~~$= \sqrt{504} = 6\sqrt{14}$~~

~~$= 6\sqrt{14} + 6\sqrt{14}$~~ ~~$= |PB|$~~

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(8) a) $\dfrac{dy}{dx} = \dfrac{\sqrt{1+2y}}{x^2}$

$\dfrac{1}{\sqrt{1+2y}}\, dy = \dfrac{1}{x^2}\, dx$

$\int (1+2y)^{-1/2}\, dy = \int x^{-2}\, dx$

$(1+2y)^{1/2} = -1/x + c$

$\sqrt{1+2y} = -1/x + c$

$y = 4,\ x = 1$

$\rightarrow \sqrt{9} = -1 + c \rightarrow c = 4$

$\therefore \sqrt{1+2y} = -1/x + 4$

b) $1 + 2y = (-1/x + 4)^2$

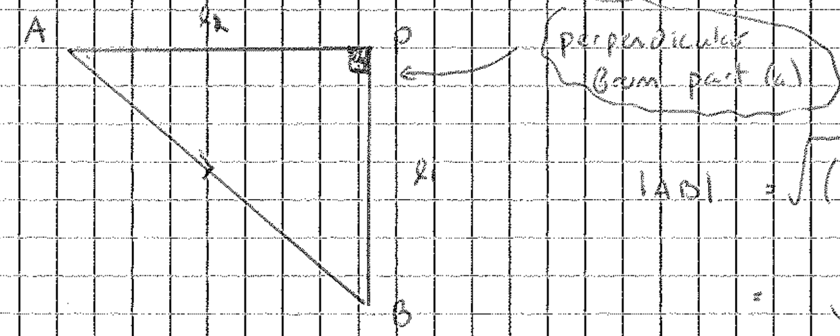
$1 + 2y = 1/x^2 - 8/x + 16$

$2y = 1/x^2 - 8/x + 15$

$y = 1/2\,(15 - 8/x + 1/x^2)$

(7) a) $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = \begin{pmatrix}4\\0\\-11\end{pmatrix} + \begin{pmatrix}2\\12\\-7\end{pmatrix} = \begin{pmatrix}6\\12\\-18\end{pmatrix}$

$|AP| = \sqrt{6^2 + 12^2 + (-18)^2} = \sqrt{504} = |PB|$



$|AB| = \sqrt{(\sqrt{504})^2 + (\sqrt{504})^2}$

$= \sqrt{2 \times 504}$

$= 12\sqrt{7}$