Core 4 - June 2006

① a) i) $p(2) = 6(2)^3 - 19(2)^2 + 9(2) + 10 = 0$

ii) $2x + 1 = 0 \rightarrow$ test $x = -\tfrac{1}{2}$

$p(-\tfrac{1}{2}) = 6(-\tfrac{1}{2})^3 - 19(-\tfrac{1}{2})^2 + 9(-\tfrac{1}{2}) + 10 = 0$

iii) $p(x) = (x-2)(2x+1)(3x-5) = 6x^3 - 19x^2 + 9x + 10$

$\uparrow$ to get $6x^3$ $\qquad \uparrow$ to get -10

b) $\dfrac{3x(x-2)}{(x-2)(2x+1)(3x-5)} = \dfrac{3x}{(2x+1)(3x-5)}$

② a) $(1-x)^{-3} = 1 + (-3)(-x) + \dfrac{(-3)(-4)}{2}(-x)^2$

$= 1 + 3x + 6x^2$

b) replace (x) with $(\tfrac{5}{2}x)$ $\rightarrow 1 + 3(\tfrac{5}{2}x) + 6(\tfrac{5}{2}x)^2$

$= 1 + \tfrac{15}{2}x + \tfrac{75}{2}x^2$

c) $|\tfrac{5}{2}x| < 1 \rightarrow |x| < \tfrac{2}{5}$

d) $\left(\dfrac{4}{2-5x}\right)^3 = \left(\dfrac{2-5x}{4}\right)^{-3} = \left(\dfrac{1}{2} - \dfrac{5}{4}x\right)^{-3} = \left(\dfrac{1}{2}\right)^{-3}\left(1 - \tfrac{5}{2}x\right)^{-3}$

$= 8\left(1 - \tfrac{5}{2}x\right)^{-3} = 8\left(1 + \tfrac{15}{2}x + \tfrac{75}{2}x^2\right)$

$= 8 + 60x + 300x^2$

③ a) $\dfrac{9x^2 - 6x + 5}{(3x-1)(x-1)} = 3 + \dfrac{A}{(3x-1)} + \dfrac{B}{(x-1)}$

$9x^2 - 6x + 5 = 3(3x-1)(x-1) + A(x-1) + B(3x-1)$

$\boxed{x = 1}$ $\quad 8 = B(2) \rightarrow B = 4$

$\boxed{x = 1/3}$ $\quad 4 = A(-\tfrac{2}{3}) \rightarrow A = -6$

b) $\displaystyle\int 3\,dx + \int \frac{-6}{3x-1}\,dx + \int \frac{4}{x-1}\,dx$

$= 3x - 2\ln(3x-1) + 4\ln(x-1) + C$

4) a) i) $\sin(2x) = 2\sin(x)\cos(x)$

ii) $\cos(2x) = 2\cos^2(x) - 1$

b) $\sin(2x) - \tan(x) = \tan(x)\cos(2x)$ Make RHS = LHS

$$2\sin(x)\cos(x) - \frac{\sin(x)}{\cos(x)} =$$

$$= \frac{2\sin(x)\cos^2(x)}{\cos(x)} - \frac{\sin(x)}{\cos(x)} = \frac{2\sin(x)\cos^2(x) - \sin(x)}{\cos(x)}$$

$$= \frac{\sin(x)}{\cos(x)}\left(2\cos^2(x) - 1\right) = \tan(x)\cos(2x)$$

c) $\sin(2x) - \tan(x) = 0 \rightarrow \tan(x)\cos(2x) = 0$

$\tan(x) = 0$

$x = 180°$

$\cos(2x) = 0$

$\cos t = 0$

$0 < x < 360$

$t = 2x$

$0 < t < 720$

$t = 90, 270, 450, 630$

$x = 45, 135, 225, 315$

$x = 45, 135, 225, 315, 180$

5) a) $y^2 - xy + 3x^2 - 5 = 0$

$\boxed{x=1} \rightarrow y^2 - y + 3 - 5 = 0$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0 \rightarrow y = 2$ and $y = -1$

b) i) $y^2 - \boxed{xy} + 3x^2 - 5 = 0$

$2y\, dy/dx - y - x\, dy/dx + 6x = 0$

$2y\, dy/dx - x\, dy/dx = y - 6x$

$dy/dx\,(2y - x) = y - 6x$

$$dy/dx = \frac{y - 6x}{2y - x}$$

$\boxed{xy}$ product rule:

$u = x \quad v = y$

$du/dx = 1 \quad dv/dx = dy/dx$

$dy/dx = y + x\, dy/dx$

ii) $x = 1,\ y = 2 \rightarrow dy/dx = \dfrac{2 - 6}{4 - 1} = -4/3$

$x = 1,\ y = -1 \rightarrow dy/dx = \dfrac{-1 - 6}{-2 - 1} = 7/3$

iii) $dy/dx = 0 \rightarrow \dfrac{y - 6x}{2y - x} = 0 \rightarrow y - 6x = 0 \rightarrow y = 6x$

SUB INTO EQUATION: $(6x)^2 - x(6x) + 3x^2 - 5 = 0$

$36x^2 - 6x^2 + 3x^2 - 5 = 0 \rightarrow 33x^2 - 5 = 0$

(6) a) i) $\vec{OB} = \begin{pmatrix}3\\2\\-1\end{pmatrix} \quad \therefore \vec{OC} = \begin{pmatrix}6\\4\\-2\end{pmatrix}$

ii) $\vec{AB} = \vec{AO} + \vec{OB} = \begin{pmatrix}-2\\-4\\-1\end{pmatrix} + \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\-2\\-2\end{pmatrix}$

b) i) $AC = \sqrt{(6-2)^2 + (4-4)^2 + (-2-1)^2}$

$= \sqrt{25} = 5$

ii) $\vec{AB} = \begin{pmatrix}1\\-2\\-2\end{pmatrix} \quad \vec{AC} = \vec{AO} + \vec{OC} = \begin{pmatrix}-2\\-4\\-1\end{pmatrix} + \begin{pmatrix}6\\4\\-2\end{pmatrix} = \begin{pmatrix}4\\0\\-3\end{pmatrix}$

$\vec{AB} \cdot \vec{AC} = \begin{pmatrix}1\\-2\\-2\end{pmatrix} \cdot \begin{pmatrix}4\\0\\-3\end{pmatrix} = 4 + 0 + 6 = 10$

$|\vec{AB}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

$|\vec{AC}| = 5 \qquad |a||b|\cos(\theta) = a.b$

$3 \times 5 \times \cos(\theta) = 10$

$\cos(\theta) = 10/15 \rightarrow \theta = \cos^{-1}(2/3) = 48.189...$
$= 48^\circ$

c) $\vec{BP} = \vec{BO} + \vec{OP} = \begin{pmatrix}-3\\-2\\1\end{pmatrix} + \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}\alpha-3\\\beta-2\\\gamma+1\end{pmatrix}$

For perpendicular, $a.b = 0 \rightarrow \begin{pmatrix}\alpha-3\\\beta-2\\\gamma+1\end{pmatrix} \cdot \begin{pmatrix}4\\0\\-3\end{pmatrix} = 0$

$\rightarrow 4(\alpha-3) + 0 - 3(\gamma+1) = 0$
$4\alpha - 12 - 3\gamma - 3 = 0$
$4\alpha - 3\gamma = 15$

(7) $dy/dx = 6xy^2 \rightarrow \int 1/y^2 \, dy = \int 6x \, dx$

$\rightarrow -1/y = 3x^2 + c$

$y=1, x=2 \rightarrow -1 = 3(2)^2 + c \rightarrow -1 = 12 + c \rightarrow c = -13$

$\rightarrow -1/y = 3x^2 - 13$

$-1 = y(3x^2 - 13)$

$\frac{-1}{3x^2-13} = y$

$y = \frac{1}{13 - 3x^2}$

8) a) i) $dx/dt \propto x(5000 - x)$ The number not infected

$dx/dt = kx(5000 - x)$

ii) $t = 0$, $dx/dt = 200$, 1000 rabbits infected

$\rightarrow 200 = k \times 1000 \times 4000$

$200 = 4{,}000{,}000k \rightarrow k = 0.00005$

b) i) $t = 4\ln\left(\frac{4 \times 2500}{5000 - 2500}\right)$

$= 4\ln(4) = 5.5451\ldots = 5.5$ hours (1dp)

ii) $30 = 4\ln\left(\frac{4x}{5000 - x}\right)$

$\rightarrow 7.5 = \ln\left(\frac{4x}{5000 - x}\right)$

$e^{7.5} = \frac{4x}{5000 - x}$

$(5000 - x)e^{7.5} = 4x$

$5000e^{7.5} - xe^{7.5} = 4x$

$5000e^{7.5} = 4x + xe^{7.5}$

$5000e^{7.5} = x(4 + e^{7.5})$

$x = \frac{5000e^{7.5}}{(4 + e^{7.5})} = 4988.96\ldots$

$= 4989$ rabbits infected