Core 4 - June 2005

(1) a) $2\sin(x) + \cos(x) = R\sin(x+\alpha)$

$= R(\sin(x)\cos(\alpha) + \cos(x)\sin(\alpha))$

$2 = R\cos(\alpha)$

$1 = R\sin(\alpha)$

$R = \sqrt{2^2+1^2} = \sqrt{5}$

$\frac{\sin(\alpha)}{\cos(\alpha)} = \frac{1}{2} \rightarrow \tan(\alpha) = 1/2 \rightarrow \alpha = 26.565\ldots$

$\rightarrow \sqrt{5}\sin(x + 26.6^\circ)$

b) $2\sin(x) + \cos(x) = 1$

$\sqrt{5}\sin(x + 26.6) = 1$

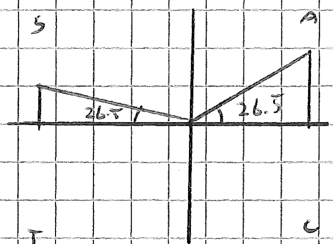
$\sin(x + 26.6) = 1/\sqrt{5}$

~~$x + 26.6 = \sin^{-1}(1/\sqrt{5})$~~

$\sin(t) = 1/\sqrt{5}$ $\qquad t = x + 26.6$

$\rightarrow x = t - 26.6$

$t = 26.565^\circ$



$t = 26.565$ or 153.434

$x = 0$ or 126.87°

(2) a) $\frac{3x-5}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$

$3x - 5 = A(2x-1) + B(x+3)$

$\boxed{x = 1/2}$ $\quad -7/2 = 7/2\,B \rightarrow B = -1$

$\boxed{x = -3}$ $\quad -14 = -7A \rightarrow A = 2$

$\frac{2}{x+3} - \frac{1}{2x-1}$

b) $\int \frac{2}{x+3} - \int \frac{1}{2x-1}$

$= 2\ln(x+3) - 1/2\ln(2x-1) + c$

(3) a) Remainder Theorem : $2x-1 \rightarrow x = 1/2$

$$f(1/2) = 2(1/2)^3 - (1/2)^2 + 2(1/2) - 2 = -1$$

b) Method 1 : $\dfrac{2x^3 - x^2 + 2x - 2}{2x-1}$

$$= \frac{x^2(2x-1)}{2x-1} + \frac{2x-2}{2x-1}$$

$$= x^2 + \frac{2x-1}{2x-1} - \frac{1}{2x-1}$$

$$= x^2 + 1 - \frac{1}{2x-1}$$

Method 2:

$$\begin{array}{r|l} & x^2 + 0x + 1 \\ \hline (2x-1) & 2x^3 - x^2 + 2x - 2 \\ & 2x^2 - x^2 \\ & \qquad\qquad 2x - 1 \end{array} \quad -1$$

$$\rightarrow x^2 + 1 - \frac{1}{2x-1}$$

(4) a) $(1+x)^{-1/2} = 1 + nx + \dfrac{n(n-1)}{2}x^2$

$$= 1 - 1/2\,x + \frac{(-1/2)(-3/2)}{2}x^2$$

$$= 1 - 1/2\,x + 3/8\,x^2$$

b) $\dfrac{1}{\sqrt{1+2x}} = (1+2x)^{-1/2}$

$$= 1 - 1/2(2x) + \frac{(-1/2)(-3/2)(2x)^2}{2}$$

$$= 1 - x + 3/2\,x^2$$

c) $x = -0.1 \rightarrow \dfrac{1}{\sqrt{1-0.2}} = \dfrac{1}{\sqrt{0.8}} = \dfrac{\sqrt{5}}{2}$

$x = -0.1 \rightarrow 1 - (-0.1) + 3/2(-0.1)^2 = 1.115$

$$\frac{\sqrt{5}}{2} \simeq 1.115$$

$$\rightarrow \sqrt{5} \simeq 1.115 \times 2 = 2.23$$

(5) a) $t = 1/2$ $\quad x = 2(1/2) + 1/{1/2} = 3$

$y = 1/{1/2} = 2 \rightarrow (3, 2)$

b) $y = 1/t$

$y = 1/t$ $\quad x = 2t + 1/t$

$t = 1/y$

$2t = 2/y$ $\quad x = 2/y + y$

$xy = 2 + y^2$

$xy - y^2 = 2$

c) $\frac{dx}{dt} = 2 - t^{-2} = 2 - \frac{1}{t^2}$

$\frac{dy}{dt} = -\frac{1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -\frac{1}{t^2} \times \frac{1}{2 - 1/t^2}$$

$$= \frac{-1}{(2t^2 - 1)}$$

At $(3, 2)$ $t = 1/2$ (from part a)

gradient: $\frac{dy}{dx} = \frac{-1}{2(1/2)^2 - 1} = 2$

(6) a) $\sin(2x) = 2\sin(x)\cos(x)$

b) i) $\cos(2x) = \cos^2(x) - \sin^2(x)$ $\quad A = B = x$

ii) $\cos(3x) = \cos(2x + x)$

$= \cos(2x)\cos(x) - \sin(2x)\sin(x)$

$= [\cos^2(x) - \sin^2(x)]\cos(x) - [2\sin(x)\cos(x)]\sin(x)$

$= \cos^3(x) - \sin^2(x)\cos(x) - 2\sin^2(x)\cos(x)$

$\sin^2(x) = 1 - \cos^2(x)$

$\cos^3(x) - (1 - \cos^2(x))(\cos(x)) - 2(1 - \cos^2(x))\cos(x)$

$\cos^3(x) - \cos(x) + \cos^3(x) - 2\cos(x) + 2\cos^3(x)$

$4\cos^3(x) - 3\cos(x)$

c) $\int_0^{\pi/2} \cos^3(x) = \int_0^{\pi/2} \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\cos(3x) + 3\cos(x) = 4\cos^3(x)$$

$$\cos^3(x) = \frac{1}{4}\cancel{4}\cos(3x) + \frac{3}{4}\cos(x)$$

$$= \left[\frac{1}{12}\sin(3x) + \frac{3}{4}\sin(x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{12}\sin(3\pi/2) + \frac{3}{4}\sin(\pi/2) - \frac{1}{12}\sin(0) - \frac{3}{4}\sin(0)$$

$$= -\frac{1}{12} + \frac{3}{4} - 0 = \frac{2}{3}$$

7 a) Distance: $\sqrt{(2-1)^2 + (-1-4)^2 + (3-2)^2}$

$$= \sqrt{1 + 25 + 1}$$

$$= \sqrt{27} = 3\sqrt{3}$$

b) $\cos(\theta) = \frac{a.b}{|a||b|}$ $\cancel{a.b = (2\times 1) + (-1\times 4) + (3\times 2) = 4}$

Vector $\vec{AB} = \begin{pmatrix} 2-1 \\ -1-4 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$

$$a.b = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 + 5 + 1 = \cancel{6}\,7$$

$$|AB| = 3\sqrt{3} \qquad \left|\begin{matrix} 1 \\ -1 \\ 1 \end{matrix}\right| = \sqrt{3}$$

$$\cos\theta = \frac{7}{3\sqrt{3}\sqrt{3}} = \frac{7}{9} \qquad \theta = \cos^{-1}(7/9)$$

c) $\vec{OP} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + p\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+p \\ -1-p \\ 3+p \end{bmatrix}$

$$\vec{AP} = \begin{bmatrix} 2+p-1 \\ -1-p-4 \\ 3+p-2 \end{bmatrix} = \begin{bmatrix} 1+p \\ -5-p \\ 1+p \end{bmatrix}$$

$$\vec{AP} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+p \\ -5-p \\ 1+p \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =$$

$$= 1 + p + 5 + p + 1 + p$$

$$= 7 + 3p$$

ii) For perpendicular, $a.b = 0$

$$7 + 3p = 0 \rightarrow p = -7/3$$

So, co-ordinates of $P = \begin{bmatrix} 2+p \\ -1-p \\ 3+p \end{bmatrix} = \begin{bmatrix} 2 - 7/3 \\ -1 + 7/3 \\ 3 - 7/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 \\ 2/3 \end{bmatrix}$

8) a) i) $t = 0 \rightarrow x = 15 + 70 = 85°c$

ii) $t = 30 \rightarrow x = 15 + 70e^{-30/40}$

$$= 48.065.. = 48°$$

iii) $60 = 15 + 70e^{-t/40}$

$45 = 70e^{-t/40}$

$45/70 = e^{-t/40}$

$\ln(45/70) = -t/40$

$-40\ln(45/70) = t = 17.67$ mins

b) i) $\frac{dx}{dt} = -\frac{1}{40}(x-15)$

$$\int \frac{1}{x-15}\,dx = \int -\frac{1}{40}\,dt$$

$$\ln(x-15) = -\frac{t}{40} + c$$

$x = 85, t = 0 \rightarrow \ln(70) = c$

$$\ln(x-15) = -\frac{t}{40} + \ln(70)$$

$$40\ln(x-15) = -t + 40\ln(70)$$

$$t = 40\ln(70) - 40\ln(x-15)$$

$$t = 40\ln\left(\frac{70}{x-15}\right)$$

ii) $t = 40 \ln\left(\frac{70}{x-15}\right)$

$\frac{t}{40} = \ln\left(\frac{70}{x-15}\right)$

$e^{\frac{t}{40}} = \frac{70}{x-15}$

$(x-15)e^{\frac{t}{40}} = 70$

$x - 15 = \frac{70}{e^{t/40}} = 70e^{-t/40}$

$x = 70e^{-t/40} + 15$