Core 3 – June 2012 Worked solutions

(1) Remember $\cot(x^2) = \frac{1}{\tan(x^2)}$

and $\int_a^b y \, dx \approx h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \dots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}\right)$ $\qquad h = \frac{b-a}{n}$

Four strips $\Rightarrow$ $n=4$ & 5 ordinates

$h = \frac{1.2-0.4}{4} = 0.2$

$x_0 = 0.4 \quad x_1 = 0.6 \quad x_2 = 0.8 \quad x_3 = 1.0 \quad x_4 = 1.2$

$x_{\frac{1}{2}} = 0.5 \quad x_{\frac{3}{2}} = 0.7 \quad x_{\frac{5}{2}} = 0.9 \quad x_{\frac{7}{2}} = 1.1$

x_i	0.5	0.7	0.9	1.1
y_i	3.9163	1.8748	0.9520	0.3773

↑ make sure your calculator is in radians!

$$\int_a^b y \, dx \approx h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + y_{\frac{5}{2}} + y_{\frac{7}{2}}\right)$$

$$= 0.2(7.1204) = 1.42408$$

$$\int_{0.4}^{1.2} \cot(x^2) \, dx \approx 1.424$$

2 (a)

At point of intersection:

$4\ln x = \sqrt{x}$ so let $f(x) = 4\ln x - \sqrt{x}$

then $f(\alpha) = 0$

$f(1.5) = 4\ln(1.5) - \sqrt{1.5} = 0.397 > 0$

$f(0.5) = 4\ln(0.5) - \sqrt{0.5} = -3.48 < 0$

Change of the sign here tells us that $f(x) = 0$ somewhere inbetween $x = 0.5$ and $x = 1.5$

so $0.5 < \alpha < 1.5$

(b) $4\ln x = \sqrt{x}$

$\ln x = \frac{\sqrt{x}}{4}$

$e^{\ln x} = e^{\sqrt{x}/4} \Rightarrow x = e^{\frac{\sqrt{x}}{4}}$

(c) $x_1 = 0.5$ $x_2 = e^{\frac{\sqrt{0.5}}{4}} = 1.193$

$x_3 = e^{\frac{\sqrt{1.193}}{4}} = 1.314$

(d) See diagram over page.

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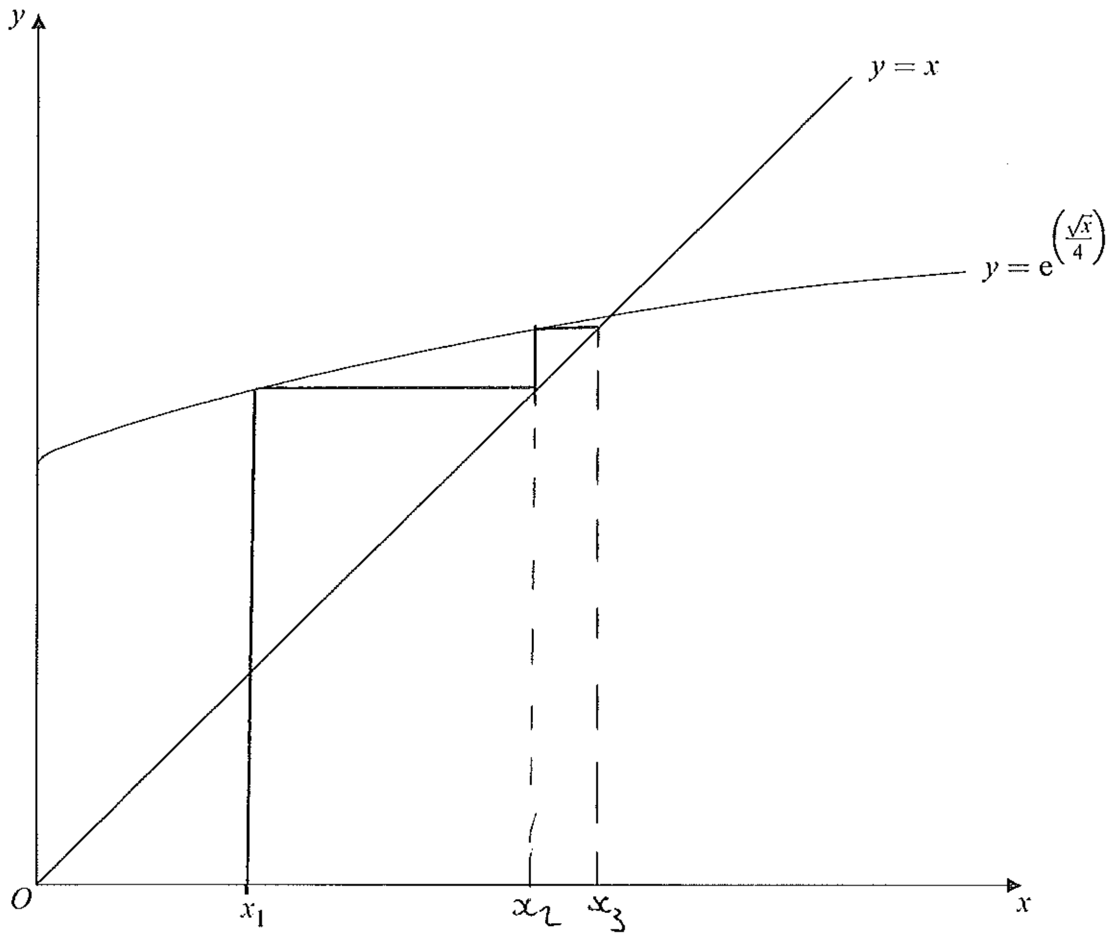
QUESTION PART REFERENCE

Answer space for question 2

(d)

Figure 1



(3) (a) Use product rule.

Let $f(x) = x^3$ $\quad g(x) = \ln x$

then $f'(x) = 3x^2$ $\quad g'(x) = \frac{1}{x}$

$$\frac{dy}{dx} = f(x)\,g'(x) + f'(x)\,g(x)$$

$$= \frac{x^3}{x} + 3x^2 \ln x$$

$$= x^2 + 3x^2 \ln x$$

(b) (i) $$\left.\frac{dy}{dx}\right|_{x=e} = e^2 + 3e^2 \ln e$$

$$= e^2 + 3e^2 = 4e^2$$

$m = 4e^2$

Need point on curve $\rightarrow$ (e, y)

$$y = x^3 \ln x$$

$$y = e^3 \ln e = e^3 \quad \text{so} \quad (e, e^3)$$

$$(y - y_1) = m(x - x_1)$$

$$y - e^3 = 4e^2(x - e)$$

(ii) $y - e^3 = 4e^2(x-e) \Rightarrow y = 4e^2(x-e) + e^3$

x-axis $\rightarrow$ $y = 0$ so $4e^2(x-e) + e^3 = 0$ (when $x = A$)

$$4e^2x - 4e^3 + e^3 = 0$$

$$4e^2x - 3e^3 = 0$$

$$4e^2x = 3e^3$$

$$x = \frac{3e^3}{4e^2} = \frac{3}{4}e$$

$$\boxed{A = \frac{3}{4}e}$$

(4) (a) Let $u = x \qquad v' = e^{6x}$

$u' = 1 \qquad v = \frac{1}{6} e^{6x}$

so $\int x e^{6x} = uv - \int u'v = \frac{1}{6} x e^{6x} - \int \frac{1}{6} e^{6x}$

$= \frac{1}{6} x e^{6x} - \frac{1}{6}\left(\frac{1}{6} e^{6x}\right) + c$

$= \frac{1}{6} x e^{6x} - \frac{1}{36} e^{6x} + c$

(b) $V = \pi \int_a^b y^2 \, dx$

$y^2 = \left(\sqrt{x}\, e^{3x}\right)^2 = x e^{6x}$

so $V = \pi \int_0^1 x e^{6x} \, dx$ (use answer to part (a)).

$V = \pi \left[\frac{1}{6} x e^{6x} - \frac{1}{36} e^{6x}\right]_0^1$

$= \pi \left[\left(\frac{1}{6} e^6 - \frac{1}{36} e^6\right) - \left(\frac{-1}{36}\right)\right]$

$= \pi \left[\frac{5}{36} e^6 + \frac{1}{36}\right]$

so $p = \frac{5}{36} \qquad q = \frac{1}{36}$

(5)

(a) $f(2.5) = \sqrt{5-5} = 0$

so $f(x) \geq 0$

(b)(i) $fg(x) = f(g(x)) = \sqrt{2\left(\frac{10}{x}\right) - 5} = \sqrt{\frac{20}{x} - 5}$

(ii) $\sqrt{\frac{20}{x} - 5} = 5$

$\frac{20}{x} - 5 = 25 \Rightarrow \frac{20}{x} = 30$

$\underline{x = \frac{20}{30} = \frac{2}{3}}$

(c)(i) $f(x)$: $y = \sqrt{2x-5}$

$f^{-1}(x)$: $x = \sqrt{2y-5}$

$x^2 = 2y - 5$

$2y = x^2 + 5$

$y = \frac{1}{2}(x^2 + 5)$

so $f^{-1}(x) = \frac{1}{2}(x^2 + 5)$

(ii) $\frac{1}{2}(x^2 + 5) = 7 \Rightarrow x^2 + 5 = 14$

$x^2 = 9 \Rightarrow x = \pm\sqrt{3}$

But range of $f(x)$ = domain of $f^{-1}(x)$

So domain of $f^{-1}(x) \geq 0$

So $\underline{x = +\sqrt{3}}$

(6) $u = x^4 + 2$

$x^4 = u - 2 \quad$ so $\boxed{x = (u-2)^{\frac{1}{4}}} \quad x^7 = (u-2)^{\frac{7}{4}}$

$\boxed{\dfrac{dx}{du} = \dfrac{1}{4}(u-2)^{-\frac{3}{4}}}$

$$\int_0^1 \frac{x^7}{(x^4+2)^2}\,dx = \int_0^1 \frac{x^7}{(x^4+2)^2}\frac{dx}{du}\,du$$

$x = 0 \Rightarrow u = 0^4 + 2 = 2 \rightarrow$ Limits

$x = 1 \Rightarrow u = 1^4 + 2 = 3$

$$\int_2^3 \frac{(u-2)^{\frac{7}{4}}}{u^2} \times \frac{1}{4}(u-2)^{-\frac{3}{4}}\,du = \frac{1}{4}\int_2^3 \frac{(u-2)^{4/4}}{u^2}\,du$$

$$= \frac{1}{4}\int_2^3 \frac{1}{u} - 2u^{-2}\,du$$

$$= \frac{1}{4}\Big[\ln u + 2u^{-1}\Big]_2^3 = \frac{1}{4}\Big[\ln 3 + \frac{2}{3} - \ln 2 - \frac{2}{2}\Big]$$

$$= \frac{1}{4}\left(\ln 3 - \ln 2 - \frac{1}{3}\right)$$

$$= \frac{1}{4}\left(\ln\frac{3}{2} - \frac{1}{3}\right) = \frac{1}{4}\ln\frac{3}{2} - \frac{1}{12}$$

$p = \frac{1}{4} \quad q = \frac{3}{2} \quad r = -\frac{1}{12}$

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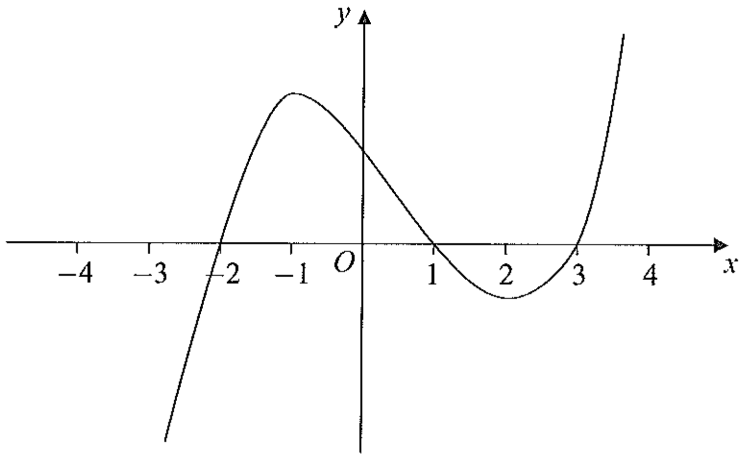
7 The sketch shows part of the curve with equation $y = \mathrm{f}(x)$.



(a) On **Figure 2** below, sketch the curve with equation $y = |\,\mathrm{f}(x)\,|$. *(3 marks)*

(b) On **Figure 3** opposite, sketch the curve with equation $y = \mathrm{f}(|\,x\,|)$. *(2 marks)*

(c) Describe a sequence of two geometrical transformations that maps the graph of $y = \mathrm{f}(x)$ onto the graph of $y = \frac{1}{2}\mathrm{f}(x+1)$. *(4 marks)*

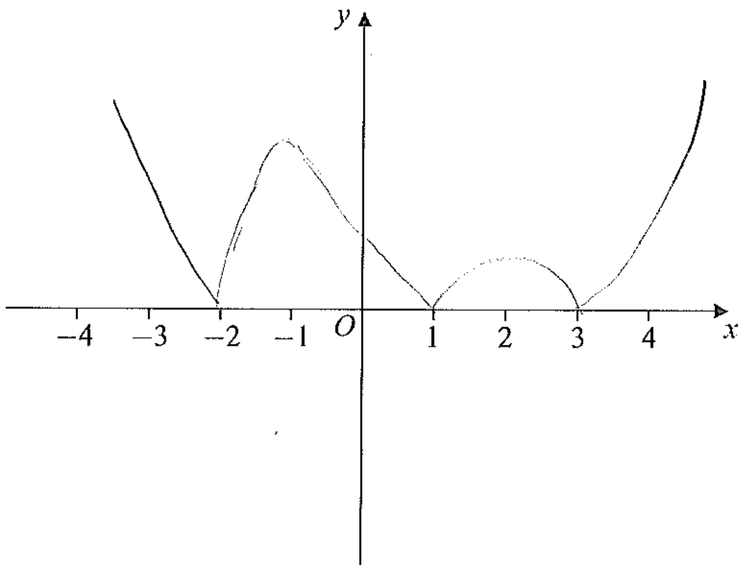
(d) The maximum point of the curve with equation $y = \mathrm{f}(x)$ has coordinates $(-1,\ 10)$. Find the coordinates of the maximum point of the curve with equation $y = \frac{1}{2}\mathrm{f}(x+1)$. *(2 marks)*

QUESTION PART REFERENCE

Answer space for question 7

(a)

Figure 2



← parts of graph where $y<0$ are reflected in x-axis

8(a) $\dfrac{1-\cos\theta}{(1+\cos\theta)(1-\cos\theta)} + \dfrac{1+\cos\theta}{(1+\cos\theta)(1-\cos\theta)} = 32$.

$\Rightarrow \dfrac{1 \cancel{-\cos\theta} + 1 \cancel{+\cos\theta}}{1-\cos^2\theta} = 32$

Since $\cos^2\theta + \sin^2\theta \equiv 1 \Rightarrow 1-\cos^2\theta = \sin^2\theta$

so $\dfrac{2}{\sin^2\theta} = 32$

$2\operatorname{cosec}^2\theta = 32$

$\operatorname{cosec}^2\theta = 16$.

(b) Let $\theta = 2x - 0\cdot6$

Then $\operatorname{cosec}^2(2x-0\cdot6) = 16$ so $\operatorname{cosec}(2x-0\cdot6) = \pm 4$

$\sin(2x-0\cdot6) = \pm\frac{1}{4}$

Then $2x - 0\cdot6 = +0\cdot253, \quad -0\cdot253$

$(+\pi) \quad +3\cdot394, \quad 2\cdot889$

so

$2x = 0\cdot853, \quad 0\cdot347, \quad 3\cdot994, \quad 3\cdot484$

$x = 0\cdot43, \quad 0\cdot17, \quad 2\cdot00, \quad 1\cdot74$

(9)

(a) $x = \frac{f(y)}{g(y)}$ then $\frac{dx}{dy} = \frac{f'(y)g(y) - f(y)g'(y)}{(g(y))^2}$

$f(y) = \sin y \qquad g(y) = \cos y$

$f'(y) = \cos y \qquad g'(y) = -\sin y$

$\frac{dx}{dy} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + \frac{\sin^2 y}{\cos^2 y} = 1 + \tan^2 y$

$1 + \tan^2 y = \sec^2 y$ so $\frac{dx}{dy} = \sec^2 y$

(b) $\tan y = x - 1$ use $1 + \tan^2 y = \sec^2 y$

$1 + \tan^2 y = 1 + (x-1)^2 = 1 + x^2 - 2x + 1 = x^2 - 2x + 2$

so $\sec^2 y = x^2 - 2x + 2$

(c) $\frac{dx}{dy} = \sec^2 y$ so $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{x^2 - 2x + 2}$

(from (a)) (from (b))

d)(i) $\frac{dy}{dx} = 0$ at stationary points

$\frac{dy}{dx} = \frac{1}{x^2 - 2x + 2} - \frac{1}{x} = 0$

(from (c))

$\Rightarrow x^2 - 2x + 2 = x$ (equating denominators)

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2, \ x = 1$

(d)(ii) $\frac{dy}{dx} = (x^2-2x+2)^{-1} - x^{-1}$

$\frac{d^2y}{dx^2} = -(x^2-2x+2)^{-2}(2x-2) + x^{-2}$

(iii) when $x = 1$

$\frac{d^2y}{dx^2} = \cancel{-(1)^{-2}(0)} + 1 = 1 > 0$

$\Rightarrow$ $x=1$ is a minimum point

when $x = 1$ $y = \tan^{-1}(1-1) - \ln(1)$

$= 0 - 0 = 0$

so $y = 0 \Rightarrow$ on x-axis.