# Core 3 Jan 2012 Worked Solutions

① (a) $\int_a^b y\,dx \approx \frac{h}{3}\left[(y_0+y_n)+4(\text{odd ordinates})+2(\text{even ordinates})\right]$

$\int_0^3 4^x\,dx \quad n=6 \quad h=\frac{3-0}{6}=0.5$

$x_0=0 \quad x_1=0.5 \quad x_2=1 \quad x_3=1.5 \quad x_4=2 \quad x_5=2.5 \quad x_6=3$

$y_0=1 \quad y_1=2 \quad y_2=4 \quad y_3=8 \quad y_4=16 \quad y_5=32 \quad y_6=64$

$\int_0^3 4^x\,dx \approx \frac{0.5}{3}\left[(1+64)+4(2+8+32)+2(4+16)\right]$

$= \frac{1}{6}\times 273 = \underline{\underline{45.5}}$

(b)(i) Let $f(x) = 4^x-(8-2x) = 4^x-8+2x$

$f(x)$ will be zero at $x=\alpha$.

$f(1.2) = 4^{1.2}-8+2(1.2) = -0.322 < 0$

$f(1.3) = 4^{1.3}-8+2(1.3) = 0.663 > 0$

Change of sign indicates solution to $f(x)=0$ lies between $x=1.2$ and $1.3$

(ii) $x_1 = 1.2 \qquad x_2 = \frac{\ln(8-2(1.2))}{\ln 4} = 1.2427$

$x_3 = \frac{\ln(8-2(1.2427))}{\ln 4} = 1.2316$

$x_2 = 1.243 \qquad \underline{\underline{x_3 = 1.232}}$

2(a) $f(1) = \frac{63}{3} = 21$  $f(16) = \frac{63}{64-1} = 1$

so $1 \leq f(x) \leq 21$

b(i) $f(x): \; y = \frac{63}{4x-1}$  so $f^{-1}(x): \; x = \frac{63}{4y-1}$

$\Rightarrow x(4y-1) = 63$

$4xy - x = 63$

$y = \frac{63+x}{4x} \Rightarrow f^{-1}(x) = \frac{63+x}{4x} = \frac{1}{4}\left(\frac{63+x}{x}\right)$

$= \frac{1}{4}\left(\frac{63}{x}+1\right)$

(ii) $\frac{1}{4}\left(\frac{63}{x}+1\right) = 1 \Rightarrow \frac{63}{x} + 1 = 4$

$\frac{63}{x} = 3 \Rightarrow x = 21$

c(i) $fg(x) = f(g(x)) = \frac{63}{4x^2-1}$

(ii) $\frac{63}{4x^2-1} = 1 \Rightarrow 63 = 4x^2 - 1$

$4x^2 = 64$

$x^2 = 16 \Rightarrow x = \pm 4$

but $g(x)$ domain : $-4 \leq x \leq -1$

so $x = -4$

3(a) $\frac{dy}{dx} = 12x^2 - 6 = 6(2x^2-1)$

(b) Use part (a) to help.

$$\frac{d}{dx}\left[\ln(4x^3 - 6x + 1)\right] = (4x^3-6x+1)^{-1}\left[6(2x^2-1)\right]$$

$$= \frac{6(2x^2-1)}{4x^3-6x+1}$$

So $$\int_2^3 \frac{1}{6}\,\frac{6(2x^2-1)}{4x^3-6x+1}\,dx = \frac{1}{6}\left[\ln(4x^3-6x+1)\right]_2^3$$

$$= \frac{1}{6}\left[\ln(4(27)-18+1) - \ln(4(8)-12+1)\right]$$

$$= \frac{1}{6}(\ln 91 - \ln 21) = \frac{1}{6}\ln\left(\frac{91}{21}\right) = \frac{1}{6}\ln\left(\frac{13}{3}\right)$$

4(a) Use $\sec^2 x = 1 + \tan^2 x \Rightarrow \tan^2\theta = \sec^2\theta - 1$

$\sec^2\theta - 1 = 9 - 3\sec\theta \Rightarrow \sec^2\theta + 3\sec\theta - 10 = 0$

Let $X = \sec\theta$

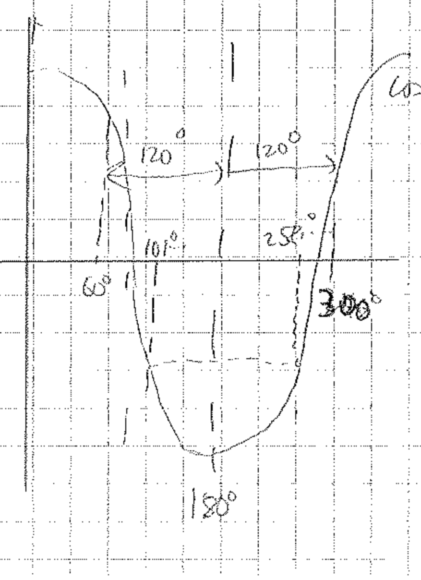
$X^2 + 3X - 10 = 0 \Rightarrow (X+5)(X-2) = 0$

$\sec\theta = -5 \quad \sec\theta = 2 \Rightarrow \cos\theta = -\frac{1}{5} \quad \cos\theta = \frac{1}{2}$

$\cos\theta = -\frac{1}{5}$ $\qquad$ $\cos\theta = \frac{1}{2}$

$\theta = 101.537°$ $\qquad$ $\theta = 60°$

$\theta = 258.463°$ $\qquad$ $\theta = 300°$




$\theta = 60°, 101.5°, 258.5°, 300°$

(Use either CAST or the fact that $\cos\theta$ is symmetrical about 180° to find the other 2 solutions)

4(b) Let $4x - 10^\circ = \theta$

then $4x - 10^\circ = 60^\circ, 101.537^\circ, 258.463^\circ, 300^\circ$

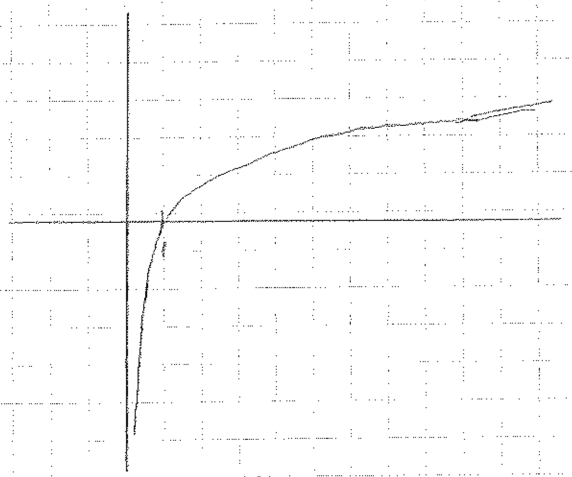
$4x = 70^\circ, 111.537^\circ, 268.463^\circ, 310^\circ$

$x = 17.5^\circ, 27.9^\circ, 67.1^\circ, 77.5^\circ$

5(a)

Translation $\begin{pmatrix} e \\ 0 \end{pmatrix}$ and a stretch with scale factor 4 in y-direction

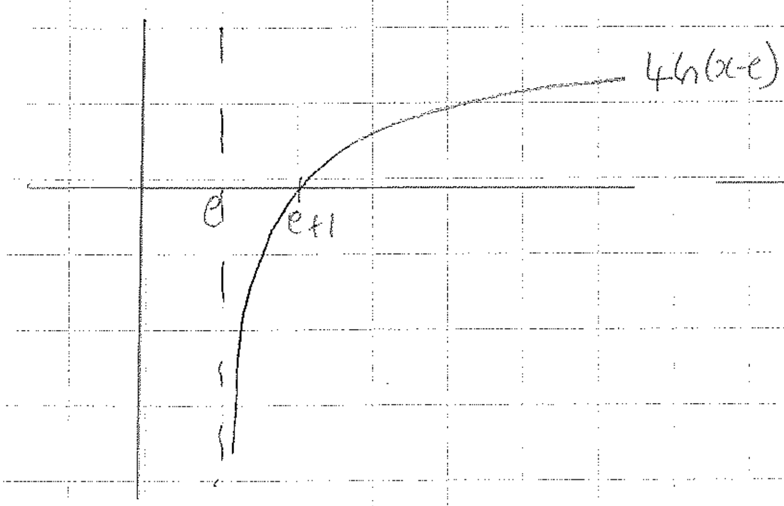
(b) Remember what $\ln x$ looks like:

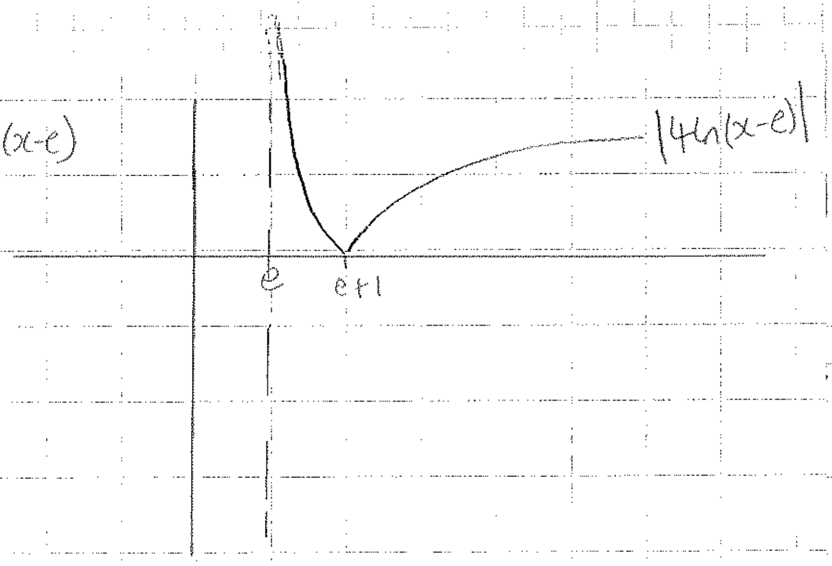


Translate by $\begin{pmatrix} e \\ 0 \end{pmatrix}$

stretch by sf 4 (since no y values shown this will not be marked.

asymptote will be at $x = e$, and x-intercept will be at $e+1$



(c) $|4\ln(x-e)| = 4$

$4\ln(x-e) = 4$ or $4\ln(x-e) = -4$

$\ln(x-e) = 1$ $\qquad$ $\ln(x-e) = -1$

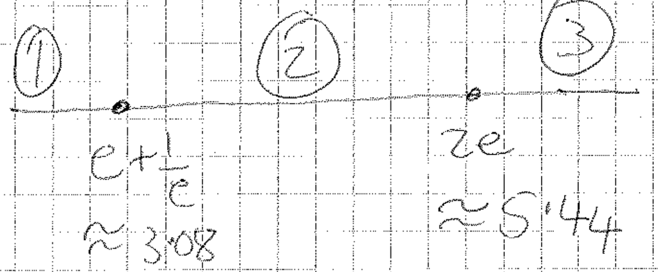
$e^{\ln(x-e)} = e^{1}$ $\qquad$ $e^{\ln(x-e)} = e^{-1}$

$x - e = e$ $\qquad$ $x - e = \frac{1}{e}$

$\underline{x = 2e}$ $\qquad$ $\underline{x = e + \frac{1}{e}}$

$|4\ln(x-e)| \geq 4$

firstly don't forget asymptote. We already know that $\underline{x > e}$.




find value of any point in region 1, 2, 3 and see if $|4\ln(x-e)| \geq 4$

e.g. 1 $x = 3$

2 $x = 4$

3 $x = 6$

You find that inequality holds in 1 and 3

so $e < x \leq e + \frac{1}{e}$, $x \geq 2e$

6

(a) $x = \frac{f(\theta)}{g(\theta)}$ $\quad \frac{dx}{d\theta} = \frac{f'(\theta)g(\theta) - f(\theta)g'(\theta)}{(g(\theta))^2}$

$f(\theta) = 1 \qquad g(\theta) = \sin\theta$

$f'(\theta) = 0 \qquad g'(\theta) = +\cos\theta$

$\frac{dx}{d\theta} = \frac{-\cos\theta}{\sin^2\theta} = \frac{-\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} = -\frac{1}{\tan\theta}\operatorname{cosec}\theta$

$= -\operatorname{cosec}\theta\cot\theta$

(b) $x = \operatorname{cosec}\theta$

$\frac{dx}{d\theta} = -\operatorname{cosec}\theta\cot\theta$

Limits

$x = 2 \Rightarrow 2 = \operatorname{cosec}\theta$

$2 = \frac{1}{\sin\theta}$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$x = \sqrt{2} \Rightarrow \sqrt{2} = \operatorname{cosec}\theta$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

$$\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1}{\operatorname{cosec}^2\theta\sqrt{\operatorname{cosec}^2\theta - 1}} \times (-\cancel{\operatorname{cosec}\theta}\cot\theta)\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{-\cancel{\cot\theta}}{\operatorname{cosec}\theta\sqrt{\cancel{\cot^2\theta}}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{-1}{\operatorname{cosec}\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} -\sin\theta\,d\theta$$

$$= \Big[\cos\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{6}} = \cos\frac{\pi}{6} - \cos\frac{\pi}{4} = \frac{\sqrt{3}-\sqrt{2}}{2} = 0.159$$

(3dp)

7 (a) Let $f(x) = x^2 \quad g(x) = e^{-\frac{x}{4}}$

Then $y = f(x)g(x)$ and $\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$

$f'(x) = 2x \quad g'(x) = -\frac{1}{4}e^{-\frac{x}{4}}$

$$\frac{dy}{dx} = 2xe^{-\frac{x}{4}} - \frac{1}{4}x^2e^{-\frac{x}{4}} = e^{-\frac{x}{4}}\left(2x - \frac{1}{4}x^2\right)$$

at stationary points $\frac{dy}{dx} = 0$

$$e^{-\frac{x}{4}}\left(2x - \frac{1}{4}x^2\right) = 0$$

$\Rightarrow e^{-\frac{x}{4}} = 0$ or $2x - \frac{1}{4}x^2 = 0$ $(x-4)$

$\underline{x = 0}$

$x^2 - 8x = 0$

$x(x-8) = 0$

$\underline{x = 8}$

If $x = 0$ $\quad x = 8$

$y = 0 \quad y = 8^2e^{-2} = \frac{64}{e^2}$

(b)(i) Just look at indefinite integral first - Ignore constant of integration as we will be calculating a definite integral

$\int x^2 e^{-\frac{x}{4}}\,dx$ let $u = x^2 \quad v' = e^{-\frac{x}{4}}$

$u' = 2x \quad v = -4e^{-\frac{x}{4}}$

$$uv - \int u'v\,dx = -4x^2e^{-\frac{x}{4}} - \int -8xe^{-\frac{x}{4}} = \boxed{-4x^2e^{-\frac{x}{4}} + \int 8xe^{-\frac{x}{4}}}$$

$\int 8xe^{-\frac{x}{4}}$ let $u = 8x \quad v' = e^{-\frac{x}{4}}$

$u' = 8 \quad v = -4e^{-\frac{x}{4}}$

$$\int 8xe^{-\frac{x}{4}} = -32xe^{-\frac{x}{4}} + \int 32e^{-\frac{x}{4}} = -32xe^{-\frac{x}{4}} - 128e^{-\frac{x}{4}}$$

$\rightarrow$

So $\int x^2 e^{-\frac{x}{4}}\,dx = -4x^2e^{-\frac{x}{4}} - 32xe^{-\frac{x}{4}} - 128e^{-\frac{x}{4}}$

$$\int_0^4 x^2 e^{-\frac{x}{4}}\,dx = \left[-4x^2e^{-\frac{x}{4}} - 32xe^{-\frac{x}{4}} - 128e^{-\frac{x}{4}}\right]_0^4$$

$$= \left[-64e^{-1} - 128e^{-1} - 128e^{-1}\right] - \left[-128\right]$$

$$= -\frac{320}{e} + 128$$

$$= 128 - \frac{320}{e}$$

b(i) $V = \pi\int_0^4 y^2\,dx$

$$= \pi\int_0^4 \left(3xe^{-\frac{x}{8}}\right)^2 dx = \pi\int_0^4 9x^2e^{-\frac{x}{4}}\,dx$$

$$= 9\pi\left[\int_0^4 x^2e^{-\frac{x}{4}}\,dx\right]$$

$$= 9\pi\left[128 - \frac{320}{e}\right]$$

January 2011 Core 3 Worked Solutions

1 (a) $y = (x^3-1)^6$ $\quad \frac{dy}{dx} = 6(x^3-1)^5 \times 3x^2$

$= 18x^2(x^3-1)^5$

(b)(i) $y = x\ln x$

Let $f(x) = x$ $\quad g(x) = \ln x$

then $y = f(x)g(x)$ $\quad$ so $\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$

$f'(x) = 1$ $\quad g'(x) = \frac{1}{x}$

$\frac{dy}{dx} = 1 + \ln x$

(ii) $\frac{dy}{dx} = 1 + \ln x$

so gradient at $x = e$ is $\ln e + 1 = 2$

$y = x\ln x$ so $x = e \Rightarrow y = e\ln e = e$

$(e, e)$ gradient $= 2$

$(y-e) = 2(x-e) \Rightarrow y - e = 2x - 2e$

$y = 2x - e$

2

(a) $y = (x^2-4)\ln(x+2)$ $\quad x \geqslant 3$

Let $f(x) = (x^2-4)\ln(x+2) - 15$

Then $f(x) = 0$ when $x = \alpha$

$f(3.5) = -0.9$

$f(3.6) = 0.4$ $\Rightarrow$ change of sign indicates $f(x) = 0$ somewhere between $x = 3.5$ and $x = 3.6$

(b) $x^2 - 4 = \frac{15}{\ln(x+2)}$ $\Rightarrow$ $x^2 = \frac{15}{\ln(x+2)} + 4$

$x = \sqrt{\frac{15}{\ln(x+2)} + 4}$

(c) $x_1 = 3.5$

$x_2 = \sqrt{4 + \frac{15}{\ln(x_1+2)}} = 3.57756$

$x_3 = \sqrt{4 + \frac{15}{\ln(x_2+2)}} = 3.56753$

$x_2 = 3.578 \quad x_3 = 3.568$

3 (a) $x = \tan(3y+1)$ $\quad \left(\frac{d}{dx}\tan x = \sec^2 x\right)$

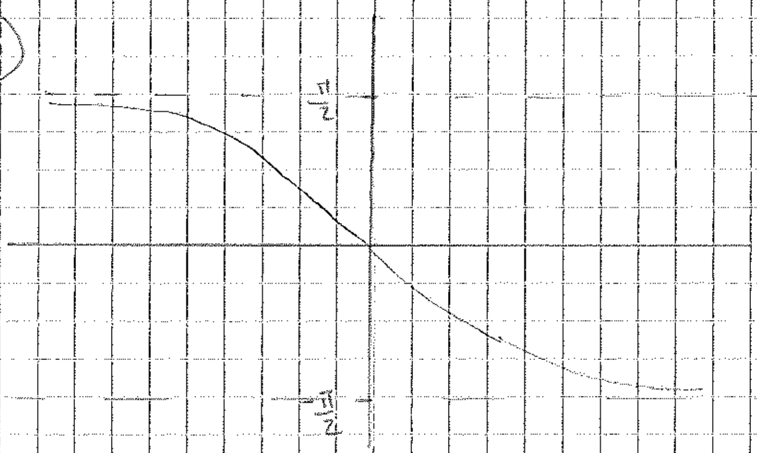
(i) $3\sec^2(3y+1)$

(ii) $y = -\frac{1}{3}$ $\quad \frac{dx}{dy} = 3\sec^2(-1+1) = 3\sec^2(0)$

$= \frac{3}{\cos^2(0)} = 3$

$\frac{dy}{dx} = \frac{1}{3}$

(b)



7 $f(x) = 3\cos\left(\frac{1}{2}x\right)$ $\qquad g(x) = |x|$

(a) $-3 \leqslant f(x) \leqslant 3$

(b)(i) $f(x): \quad y = 3\cos\left(\frac{1}{2}x\right)$

$f^{-1}(x): \quad x = 3\cos\left(\frac{1}{2}y\right)$

$\frac{x}{3} = \cos\left(\frac{1}{2}y\right)$

$\frac{1}{2}y = \cos^{-1}\left(\frac{x}{3}\right)$

$y = 2\cos^{-1}\left(\frac{x}{3}\right)$

so $\quad f^{-1}(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

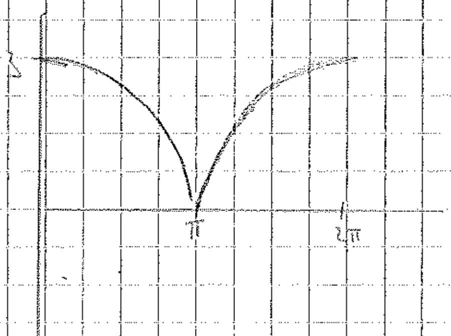
(ii) $f^{-1}(x) = 1 \qquad 2\cos^{-1}\left(\frac{x}{3}\right) = 1$

$\cos^{-1}\left(\frac{x}{3}\right) = \frac{1}{2}$

$\cos\frac{1}{2} = \frac{x}{3}$

$x = 3\cos\left(\frac{1}{2}\right)$

(c)(i) $gf(x) = g(f(x)) = \left|3\cos\frac{1}{2}x\right|$




(ii) Stretch with scale factor 3 in y-direction
& Stretch with scale factor 2 in x-direction.

7(a) $\displaystyle\int \frac{1}{3+2x}\,dx$

$\displaystyle\frac{d}{dx}\ln(3+2x) = \frac{1}{3+2x} \times 2 = \frac{2}{3+2x}$

$\displaystyle\int = \frac{1}{2}\int \frac{2}{3+2x}\,dx = \frac{1}{2}\ln(3+2x) + C$

(b) $\displaystyle\int x \sin\frac{x}{2}\,dx$ let $u = x$ $v' = \sin\frac{x}{2}$

$u' = 1$ $v = -2\cos\frac{x}{2}$

$\displaystyle uv - \int u'v = -2x\cos\frac{x}{2} + \int 2\cos\frac{x}{2}\,dx$

$\displaystyle = -2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + C$

6(a) $\displaystyle h = \frac{0.4-0}{4} = 0.1$

$x_0 = 0$ $x_1 = 0.1$ $x_2 = 0.2$ $x_3 = 0.3$ $x_4 = 0.4$

$x_{\frac{1}{2}} = 0.05$ $x_{\frac{3}{2}} = 0.15$ $x_{\frac{5}{2}} = 0.25$ $x_{\frac{7}{2}} = 0.35$

$y_{\frac{1}{2}} = 0.4780$ $y_{\frac{3}{2}} = 0.3585$ $y_{\frac{5}{2}} = 0.2454$ $y_{\frac{7}{2}} = 0.1386$

$\displaystyle\int \approx h\left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + y_{\frac{5}{2}} + y_{\frac{7}{2}}\right] = 0.12205$

$= 0.122$ (3dp).

(b) $u = 3x + 1$ $\qquad \frac{du}{dx} = 3$ $\qquad \boxed{\frac{dx}{du} = \frac{1}{3}}$

$x = \frac{u-1}{3}$

Limits:

$x = 0 \qquad u = 1$

$x = 1 \qquad u = 4$

$$\int_0^1 x\sqrt{3x+1}\,dx = \int_1^4 \frac{u-1}{3}\sqrt{u} \times \frac{1}{3}\,du$$

$$= \frac{1}{9}\int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \frac{1}{9}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{9}\left[\left(\frac{2}{5}4^{\frac{5}{2}} - \frac{2}{3}4^{\frac{3}{2}}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right]$$

$$= \frac{1}{9} \times \frac{116}{15} = \frac{116}{135}$$

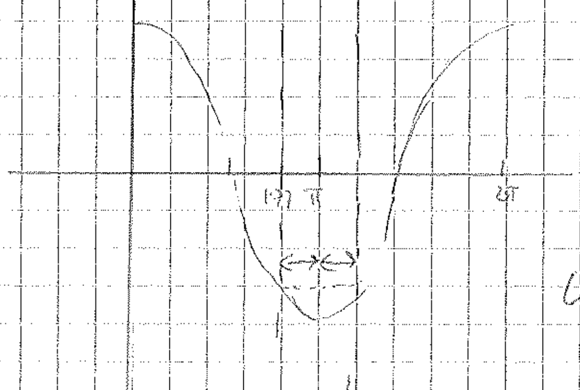
7(a) $\sec x = -5$

$\frac{1}{\cos x} = -5$

$\cos x = -\frac{1}{5}$

$x = 1.77$

$x = 4.5$




$\cos x$ is symmetrical about $x = \pi$.

(b) Use common denominator $(1+\operatorname{cosec}x)(1-\operatorname{cosec}x) = 1-\operatorname{cosec}^2 x$

$$\frac{\operatorname{cosec}x}{(1+\operatorname{cosec}x)} \times \frac{(1-\operatorname{cosec}x)}{(1-\operatorname{cosec}x)} = \frac{\operatorname{cosec}x\,(1-\operatorname{cosec}x)}{1-\operatorname{cosec}^2 x}$$

$$\frac{\operatorname{cosec}x}{(1-\operatorname{cosec}x)} \times \frac{(1+\operatorname{cosec}x)}{(1+\operatorname{cosec}x)} = \frac{\operatorname{cosec}x\,(1+\operatorname{cosec}x)}{1-\operatorname{cosec}^2 x}$$

$$\frac{\cancel{\operatorname{cosec}x} - \operatorname{cosec}^2 x - \cancel{\operatorname{cosec}x} - \operatorname{cosec}^2 x}{1-\operatorname{cosec}^2 x}$$

$$= \frac{-2\operatorname{cosec}^2 x}{1-\operatorname{cosec}^2 x}$$

Use $\operatorname{cosec}^2 A = 1+\cot^2 A$

$1-\operatorname{cosec}^2 A = -\cot^2 A$

$$= \frac{-2\operatorname{cosec}^2 x}{-\cot^2 x}$$

$$= \frac{2\tan^2 x}{\sin^2 x}$$

Use $\tan x = \dfrac{\sin x}{\cos x}$

$$= \frac{2\cancel{\sin^2 x}}{\cos^2 x\,\cancel{\sin^2 x}} = \frac{2}{\cos^2 x} = 2\sec^2 x$$

so $2\sec^2 x = 50 \Rightarrow \underline{\sec^2 x = 25}$

(c) $\sec^2 x = 25$

$\sec x = \pm 5$

$\sec x = -5$

$x = 1.77$

$x = 4.51$ (from (a))

$\sec x = 5$

$\cos x = \frac{1}{5}$

$x = 1.37$

$x = 4.91$

either CAST or symmetry of $\cos x$ graph about $x = \pi$

3(a) $e^{-2x} = 4$

$\ln e^{-2x} = \ln 4$

$-2x = \ln 4 \Rightarrow x = -\frac{1}{2}\ln 4$

$= \ln 4^{-\frac{1}{2}} = \ln \frac{1}{2}$

(b) (i) $x = 0$

$y = 4 - 1 = 3$

(ii) $y = 0$

$4e^{-2x} - e^{-4x} = 0$

$4e^{-2x} = e^{-4x}$

$4 = e^{-4x} \div e^{-2x} = e^{-2x}$

$\ln 4 = \ln e^{-2x} \Rightarrow \ln 4 = -2x$

$x = -\frac{1}{2}\ln 4 = \ln\frac{1}{2}$

(iii) $\frac{dy}{dx} = -8e^{-2x} + 4e^{-4x}$

at stationary points $\frac{dy}{dx} = 0$

$-8e^{-2x} + 4e^{-4x} = 0$

$(\times e^{4x})$ $-8e^{2x} + 4 = 0$

$4 = 8e^{2x}$

$e^{2x} = \frac{1}{2}$

$\ln e^{2x} = \ln \frac{1}{2}$

$2x = \ln \frac{1}{2}$

$x = \frac{1}{2}\ln\frac{1}{2}$

(iv) $\pi\int_0^{\ln 2} y^2\,dx$

$$y^2 = (4e^{-2x} - e^{-4x})(4e^{-2x} - e^{-4x})$$

$$= 16e^{-4x} - 8e^{-6x} + e^{-8x}$$

$$\pi\int_0^{\ln 2} 16e^{-4x} - 8e^{-6x} + e^{-8x}\,dx$$

$$= \pi\left[-4e^{-4x} + \frac{4}{3}e^{-6x} - \frac{1}{8}e^{-8x}\right]_0^{\ln 2}$$

$$= \pi\left[\left(-4e^{-4\ln 2} + \frac{4}{3}e^{-6\ln 2} - \frac{1}{8}e^{-8\ln 2}\right) - \left(-4 + \frac{4}{3} - \frac{1}{8}\right)\right]$$

$$= \pi\left[-4(2^{-4}) + \frac{4}{3}(2^{-6}) - \frac{1}{8}(2^{-8}) + \frac{67}{24}\right]$$

$$= \pi\left[-\frac{1471}{6144} + \frac{67}{24}\right] = \frac{5247}{2048}\pi$$