June 2011 Core 3 Worked Solutions

(1) (a) $y = \ln(6x)$

$y=0 \Rightarrow \ln(6x) = 0$ $\quad (\ln 1 = 0)$

$6x = 1$

$x = \frac{1}{6}$

(b) $\frac{d}{dx} \ln(6x) = \frac{6}{6x} = \frac{1}{x}$

(c) $h = \frac{6}{6} = 1$ $\quad$ $\left(\text{using: } h = \frac{b-a}{n}\right)$

$x_0 = 1 \quad x_1 = 2 \quad x_2 = 3 \quad x_3 = 4 \quad x_4 = 5 \quad x_5 = 6 \quad x_6 = 7$

$y_0 = 1.7918 \quad y_1 = 2.4849 \quad y_2 = 2.8904 \quad y_3 = 3.1781 \quad y_4 = 3.4012 \quad y_5 = 3.5835$

$y_6 = 3.7377$

$$\int \approx \frac{1}{3} \times 1 \left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$= 18.4$ (3sf)

(2)(a)(i) Let $f(x) = x \qquad g(x) = e^{2x}$ $\qquad$ then $y = f(x)g(x)$

$f'(x) = 1 \qquad g'(x) = 2e^{2x}$

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x) = e^{2x} + 2xe^{2x}$$

(ii) $m = e^{2x} + 2xe^{2x}$ $\quad$ when $x = 1$ $\quad m = e^2 + 2e^2 = 3e^2$

$y - e^2 = 3e^2(x-1)$

(b) $y = \dfrac{2\sin 3x}{1+\cos 3x}$ let $f(x) = 2\sin 3x$ $\quad g(x) = 1+\cos 3x$

$f'(x) = 6\cos 3x$ $\quad g'(x) = -3\sin 3x$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{6\cos 3x(1+\cos 3x) + 6\sin^2 3x}{(1+\cos 3x)^2}$$

$$= \frac{6\cos 3x + 6\cos^2 3x + 6\sin^2 3x}{(1+\cos 3x)^2}$$

(Use $\sin^2 A + \cos^2 A = 1$)

$$= \frac{6\cos 3x + 6}{(1+\cos 3x)^2} = \frac{6\cancel{(1+\cos 3x)}}{(1+\cos 3x)^{\cancel{2}}} = \frac{6}{1+\cos 3x}$$

$\underline{\underline{k = 6}}$

(3) (a) let $f(x) = \cos^{-1}(2x-1) - e^{x}$

$f(0.4) = 0.3$

$f(0.5) = -0.08$

$\Rightarrow$ change of sign indicates that solution to $f(x)=0$ lies between $x=0.4$ and $x=0.5$

(b) $e^{x} = \cos^{-1}(2x-1)$

$\cos(e^{x}) = 2x-1$

$1+\cos(e^{x}) = 2x$

$x = \frac{1}{2}\left(1+\cos(e^{x})\right)$

(c) $x_1 = 0.4$

$x_2 = \frac{1}{2}\left(1+\cos(e^{0.4})\right) = 0.539$

$x_3 = \frac{1}{2}\left(1+\cos(e^{x_2})\right) = 0.428$

when substituting in x_2 do not use rounded value.

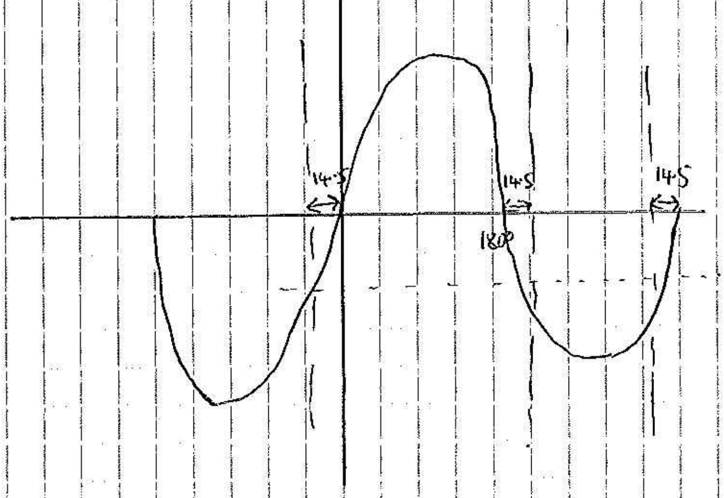
4 (a) (i) $\frac{1}{\sin\theta} = -4 \qquad \sin\theta = \frac{-1}{4}$

$\theta = -14.5^{\circ}$

194.5°

345.5°

(either use CAST or symmetry of sin graph)



(ii) $2\cot^2(2x+30) = 2 - 7\operatorname{cosec}(2x+30°)$

use $\operatorname{cosec}^2 A = 1 + \cot^2 A$

$\cot^2 A = \operatorname{cosec}^2 A - 1$

$2\operatorname{cosec}^2(2x+30) - 2 = 2 - 7\operatorname{cosec}(2x+30)$

$2\operatorname{cosec}^2(2x+30) + 7\operatorname{cosec}(2x+30) - 4 = 0$

let $X = \operatorname{cosec}(2x+30)$

$2X^2 + 7X - 4 = 0$

$(2X-1)(X+4) = 0$

$X = \frac{1}{2}$

$\operatorname{cosec}(2x+30) = \frac{1}{2}$

$\sin(2x+30) = 2$ X no solution

$X = -4$

$\operatorname{cosec}(2x+30) = -4$

$\sin(2x+30) = \frac{-1}{4}$

$2x+30 = 194.5$

345.5

$x = 82.5°$

and $157.8°$

4(b) Translation by $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$

and then a stretch with scale factor $\frac{1}{2}$ in x direction.

(5) $f(x) = x^2$

If $f^{-1}(x)$ exists then:

$f(x): \quad y = x^2$

$f^{-1}(x): \quad x = y^2 \implies y = \sqrt{x}$

but $f^{-1}(x) = \sqrt{x}$ is not one to one.

eg if $x = 4$

$f^{-1}(x) = +2$ and -2.

(b) $g(x) = \dfrac{1}{2x+1}$

$$y = \frac{1}{2x+1}$$

$g^{-1}(x): \quad x = \dfrac{1}{2y+1} \implies 2xy + x = 1$

$$y = \frac{1-x}{2x}$$

$$g^{-1}(x) = \frac{1}{2x} - \frac{x}{2x} = \frac{1}{2}\left(\frac{1}{x} - 1\right)$$

(c) range of $g^{-1}(x)$ = domain of $g(x)$

so: range is all real values

$y = x, \quad x \neq -0.5$

(d) $fg(x) = \left(\dfrac{1}{2x+1}\right)^2$

$fg(x) = g(x) \implies \left(\dfrac{1}{2x+1}\right)^2 = \dfrac{1}{2x+1} \quad (1^2 = 1)$

$\dfrac{1}{2x+1} = 1 \implies 2x + 1 = 0$

$\implies \boxed{x = 0}$

6(a) $3\ln x = 4$

$\ln x^3 = 4$

$e^{\ln x^3} = e^4$ so $x^3 = e^4$

$x = \sqrt[3]{e^4} = e^{\frac{4}{3}}$

(b) $3\ln x + \frac{20}{\ln x} = 19$

$(\times \ln x)$ $3(\ln x)^2 + 20 = 19\ln x$

$3(\ln x)^2 - 19\ln x + 20 = 0$

let $X = \ln x$

$3X^2 - 19X + 20 = 0$

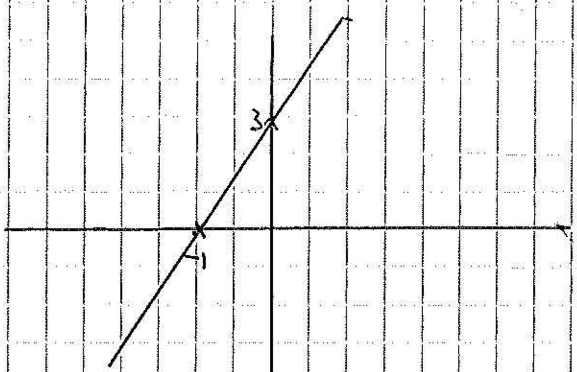
$(3X - 4)(X - 5) = 0$

$X = \frac{4}{3}$ $\quad X = 5$

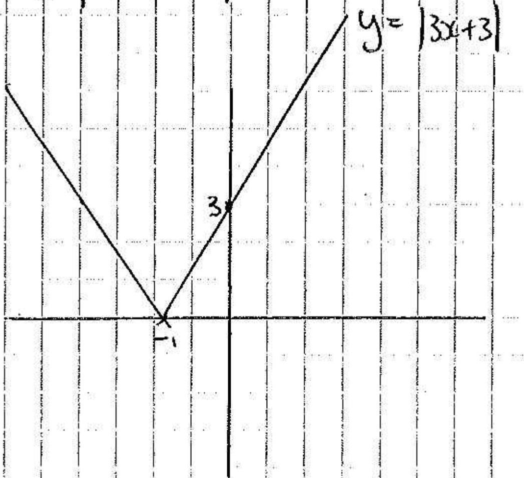
$\ln x = \frac{4}{3}$ $\quad \ln x = 5$

$\underline{x = e^{\frac{4}{3}}}$ $\quad \underline{x = e^5}$

7(a)(i) Consider $y = 3x + 3$ first :

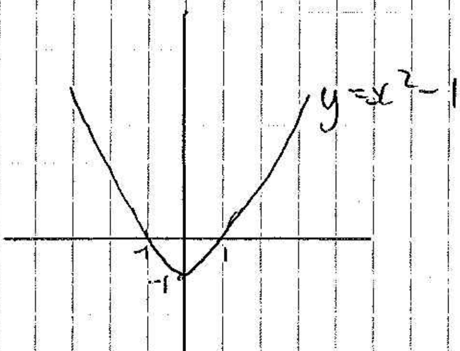


so $y = |3x + 3|$ is :

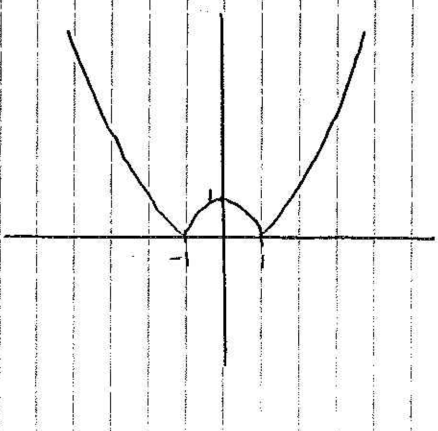


(ii) $y = x^2 - 1$

$= (x+1)(x-1)$



$y = |x^2 - 1|$



(b)(i) $|3x+3| = |x^2-1|$

$3x+3 = x^2-1$ | $3x+3 = -(x^2-1)$

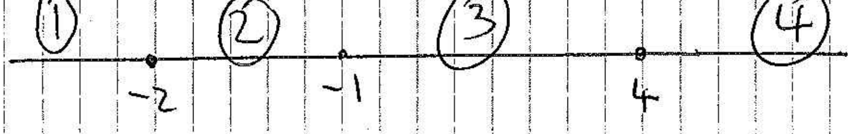
$x^2-3x-4=0$ | $x^2+3x+2=0$

$(x-4)(x+1)=0$ | $(x+2)(x+1)=0$

$\underline{x=4}$, $\underline{x=-1}$ | $\underline{x=-2}$ $x=-1$

$$x = -1, -2, 4$$

(ii)



Check if $|3x+3| < |x^2-1|$ applies in region 1, 2, 3 and 4

Checking one point in each region is enough.

1 use $x=-3$ ✓

2 use $x=-1.5$ ✗

3 use $x=0$ ✗

4 use $x=5$ ✓

so 1 $x<-2$

4 $x>4$

(8) $u = 1 + 2\tan x$

$$\frac{du}{dx} = 2\sec^2 x$$

so $$\frac{dx}{du} = \frac{1}{2\sec^2 x} = \frac{1}{2}\cos^2 x$$

$$\int \frac{1}{u^2 \cancel{\cos^2 x}} \times \frac{1}{2}\cancel{\cos^2 x} \quad du$$

$$= \frac{1}{2}\int \frac{1}{u^2}\, du = \frac{1}{2}\int u^{-2}\, du$$

$$= \frac{1}{2}\left[-1\, u^{-1}\right] + C$$

$$= \frac{-1}{2} \times \frac{1}{u} + C$$

$$= \frac{-1}{2}\,\frac{1}{1+2\tan x} + C$$

$$= \frac{-1}{2+4\tan x} + C$$

(9) (a) let $u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{1}{2}x^2$

$$uv - \int u'v = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

(b) $y = (\ln x)^2 = (\ln x)(\ln x)$

$f(x) = \ln x \qquad g(x) = \ln x \qquad f'(x) = g'(x) = \frac{1}{x}$

$$\frac{dy}{dx} = 2f'(x)f(x) \qquad \left(\text{product rule when } f(x) = g(x)\right)$$

$$= 2\frac{1}{x}\ln x$$

$$= \frac{2}{x}\ln x$$

(c) $\pi \int_1^e y^2 \, dx \qquad y^2 = x(\ln x)^2$

$\pi \int_1^e x(\ln x)^2 \, dx \qquad$ let $u = (\ln x)^2 \qquad v' = x$

$u' = \frac{2}{x}\ln x \qquad v = \frac{1}{2}x^2$

$$uv - \int u'v = \frac{1}{2}x^2(\ln x)^2 - \int \frac{\cancel{2}}{\cancel{x}} \times \frac{1}{\cancel{2}}x^{\cancel{2}} \ln x$$

$$= \frac{1}{2}x^2(\ln x)^2 - \int x \ln x \qquad \text{(now use part (a))}$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{x^2}{4} + C$$

$$\pi \int_1^e x(\ln x)^2 \, dx = \pi\left[\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{x^2}{4}\right]_1^e$$

$$= \pi\left[\left(\cancel{\tfrac{1}{2}e^2} - \cancel{\tfrac{1}{2}e^2} + \frac{e^2}{4}\right) - \left(\frac{1}{4}\right)\right]$$

$$= \pi\left(\frac{e^2}{4} - \frac{1}{4}\right) = \underline{\underline{\frac{\pi}{4}\left(e^2 - 1\right)}}$$