January 2011 Core 3 Worked Solutions

(a) $y = (x^3 - 1)^6$ $\qquad \frac{dy}{dx} = 6(x^3 - 1)^5 \times 3x^2$

$$= 18x^2(x^3 - 1)^5$$

(b)(i) $y = x\ln x$

Let $f(x) = x$ $\qquad g(x) = \ln x$

then $y = f(x)g(x)$ $\qquad$ so $\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$

$f'(x) = 1$ $\qquad g'(x) = \frac{1}{x}$

$$\frac{dy}{dx} = 1 + \ln x$$

(ii) $\frac{dy}{dx} = 1 + \ln x$

so gradient at $x = e$ is $\ln e + 1 = 2$

$y = x\ln x$ so $x = e \Rightarrow y = e\ln e = e$

(e, e) $\qquad$ gradient $= 2$

$(y - e) = 2(x - e) \Rightarrow y - e = 2x - 2e$

$y = 2x - e$

2

(a) $y = (x^2 - 4)\ln(x + 2)$ $\qquad x \geq 3$

Let $f(x) = (x^2 - 4)\ln(x + 2) - 15$

Then $f(x) = 0$ when $x = \alpha$

$f(3.5) = -0.9$

$f(3.6) = 0.4$ $\Rightarrow$ change of sign indicates $f(x) = 0$ somewhere between $x = 3.5$ and $x = 3.6$

(b) $x^2 - 4 = \frac{15}{\ln(x+2)}$ $\Rightarrow x^2 = \frac{15}{\ln(x+2)} + 4$

$$x = \sqrt{\frac{15}{\ln(x+2)} + 4}$$

(c) $x_1 = 3.5$

$$x_2 = \sqrt{4 + \frac{15}{\ln(x_1+2)}} = 3.57756$$

$$x_3 = \sqrt{4 + \frac{15}{\ln(x_2+2)}} = 3.56753$$

$x_2 = 3.578$ $\quad$ $x_3 = 3.568$

3(a) $x = \tan(3y+1)$ $\quad$ $\left(\frac{d}{dx}\tan x = \sec^2 x\right)$

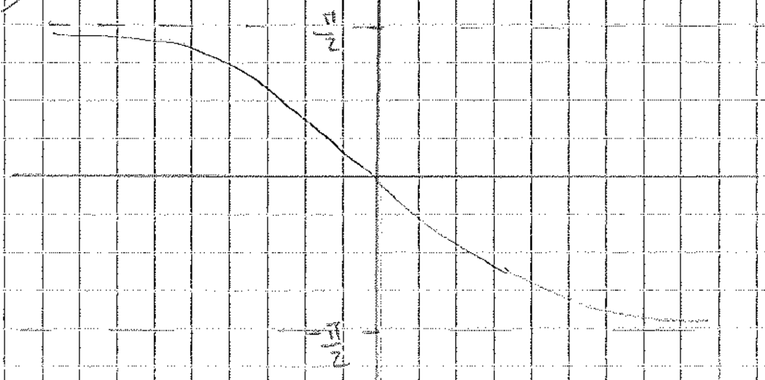
(i) $3\sec^2(3y+1)$

(ii) $y = \frac{-1}{3}$ $\quad$ $\frac{dx}{dy} = 3\sec^2(-1+1) = 3\sec^2(0)$

$$= \frac{3}{\cos^2(0)} = 3$$

$$\frac{dy}{dx} = \frac{1}{3}$$

(b)



1 $f(x) = 3\cos\left(\frac{1}{2}x\right)$ $\quad g(x) = |x|$

(a) $-3 \leqslant f(x) \leqslant 3$

(b)(i) $f(x)$: $y = 3\cos\left(\frac{1}{2}x\right)$

$f^{-1}(x)$: $x = 3\cos\left(\frac{1}{2}y\right)$

$\frac{x}{3} = \cos\left(\frac{1}{2}y\right)$

$\frac{1}{2}y = \cos^{-1}\left(\frac{x}{3}\right)$

$y = 2\cos^{-1}\left(\frac{x}{3}\right)$

so $f^{-1}(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

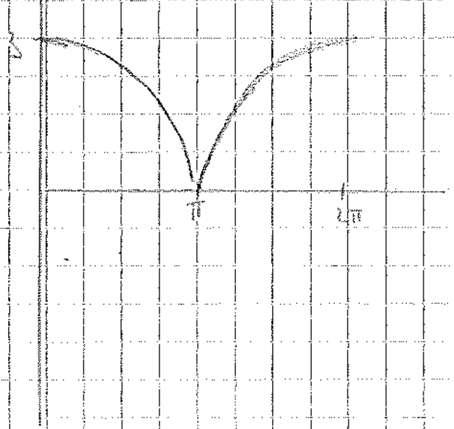
(ii) $f^{-1}(x) = 1$ $\quad 2\cos^{-1}\left(\frac{x}{3}\right) = 1$

$\cos^{-1}\left(\frac{x}{3}\right) = \frac{1}{2}$

$\cos\frac{1}{2} = \frac{x}{3}$

$x = 3\cos\left(\frac{1}{2}\right)$

(c)(i) $gf(x) = g(f(x)) = \left|3\cos\frac{1}{2}x\right|$



(ii) Stretch with scale factor 3 in y-direction
& stretch with scale factor 2 in x-direction.

a) $\int \frac{1}{3+2x}\,dx$

$\frac{d}{dx}\ln(3+2x) = \frac{1}{3+2x}\times 2 = \frac{2}{3+2x}$

$\int = \frac{1}{2}\int \frac{2}{3+2x}\,dx = \frac{1}{2}\ln(3+2x) + C$

b) $\int x \sin\frac{x}{2}\,dx$ let $u = x$ $\quad v' = \sin\frac{x}{2}$

$u' = 1 \quad v = -2\cos\frac{x}{2}$

$uv - \int u'v = -2x\cos\frac{x}{2} + \int 2\cos\frac{x}{2}\,dx$

$= -2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + C$

5(a) $h = \frac{0.4-0}{4} = 0.1$

$x_0 = 0 \quad x_1 = 0.1 \quad x_2 = 0.2 \quad x_3 = 0.3 \quad x_4 = 0.4$

$x_{\frac{1}{2}} = 0.05 \quad x_{\frac{3}{2}} = 0.15 \quad x_{\frac{5}{2}} = 0.25 \quad x_{\frac{7}{2}} = 0.35$

$y_{\frac{1}{2}} = 0.4780 \quad y_{\frac{3}{2}} = 0.3885 \quad y_{\frac{5}{2}} = 0.2454 \quad y_{\frac{7}{2}} = 0.1386$

$\int \approx h\left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + y_{\frac{5}{2}} + y_{\frac{7}{2}}\right] = 0.122205$

$= 0.122$ (3 dp).

(b) $u = 3x + 1$ $\qquad \frac{du}{dx} = 3 \qquad \boxed{\frac{dx}{du} = \frac{1}{3}}$

$x = \frac{u-1}{3}$

Limits:

$x = 0 \qquad u = 1$

$x = 1 \qquad u = 4$

$$\int_0^1 x\sqrt{3x+1}\,dx = \int_1^4 \frac{u-1}{3}\sqrt{u} \times \frac{1}{3}\,du$$

$$= \frac{1}{9}\int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \frac{1}{9}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{9}\left[\left(\frac{2}{5}4^{\frac{5}{2}} - \frac{2}{3}4^{\frac{3}{2}}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right]$$

$$= \frac{1}{9} \times \frac{116}{15} = \frac{116}{135}$$

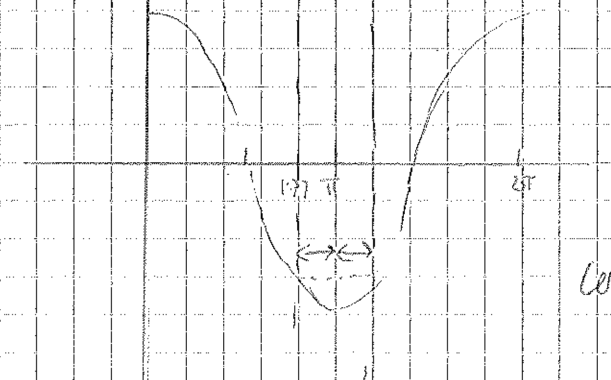
7(a) $\sec x = -5$

$\frac{1}{\cos x} = -5$

$\cos x = -\frac{1}{5}$

$x = 1.77$

$x = 4.5$



$\cos x$ is symmetrical about $x = \pi$.

(b) Use common denominator $(1+\operatorname{cosec} x)(1-\operatorname{cosec} x) = 1-\operatorname{cosec}^2 x$

$$\frac{\operatorname{cosec} x}{(1+\operatorname{cosec} x)} \times \frac{(1-\operatorname{cosec} x)}{(1-\operatorname{cosec} x)} = \frac{\operatorname{cosec} x\,(1-\operatorname{cosec} x)}{1-\operatorname{cosec}^2 x}$$

$$\frac{\operatorname{cosec} x}{(1-\operatorname{cosec} x)} \times \frac{(1+\operatorname{cosec} x)}{(1+\operatorname{cosec} x)} = \frac{\operatorname{cosec} x\,(1+\operatorname{cosec} x)}{1-\operatorname{cosec}^2 x}$$

$$\frac{\cancel{\operatorname{cosec} x} - \operatorname{cosec}^2 x - \cancel{\operatorname{cosec} x} - \operatorname{cosec}^2 x}{1-\operatorname{cosec}^2 x}$$

$$= \frac{-2\operatorname{cosec}^2 x}{1-\operatorname{cosec}^2 x}$$

Use $\operatorname{cosec}^2 A = 1+\cot^2 A$

$1-\operatorname{cosec}^2 A = -\cot^2 A$

$$= \frac{-2\operatorname{cosec}^2 x}{-\cot^2 x}$$

$$= \frac{2\tan^2 x}{\sin^2 x}$$

Use $\tan x = \dfrac{\sin x}{\cos x}$

$$= \frac{2\cancel{\sin^2 x}}{\cos^2 x\,\cancel{\sin^2 x}} = \frac{2}{\cos^2 x} = 2\sec^2 x$$

so $2\sec^2 x = 50 \Rightarrow \underline{\sec^2 x = 25}$

(c) $\sec^2 x = 25$

$\sec x = \pm 5$

$\sec x = -5$

$x = 1.77$ (from (a))

$x = 4.51$

$\sec x = 5$

$\cos x = \frac{1}{5}$

$x = 1.37$

$x = 4.91$

either CAST or symmetry of $\cos x$ graph about $x=\pi$

3(a) $e^{-2x} = 4$

$\ln e^{-2x} = \ln 4$

$-2x = \ln 4 \Rightarrow x = -\frac{1}{2}\ln 4$

$= \ln 4^{-\frac{1}{2}} = \ln \frac{1}{2}$

(b) (i) $x = 0$

$y = 4 - 1 = \underline{3}$

(ii) $y = 0$

$4e^{-2x} - e^{-4x} = 0$

$4e^{-2x} = e^{-4x}$

$4 = e^{-4x} \div e^{-2x} = e^{-2x}$

$\ln 4 = \ln e^{-2x} \Rightarrow \ln 4 = -2x$

$x = -\frac{1}{2}\ln 4 = \ln \frac{1}{2}$

(iii) $\frac{dy}{dx} = -8e^{-2x} + 4e^{-4x}$

at stationary points $\frac{dy}{dx} = 0$

$-8e^{-2x} + 4e^{-4x} = 0$

$(\times e^{4x})$ $-8e^{2x} + 4 = 0$

$4 = 8e^{2x}$

$e^{2x} = \frac{1}{2}$

$\ln e^{2x} = \ln \frac{1}{2}$

$2x = \ln \frac{1}{2}$

$\underline{x = \frac{1}{2}\ln \frac{1}{2}}$

(iv) $\pi \int_0^{\ln 2} y^2 \, dx$

$y^2 = \left(4e^{-2x} - e^{-4x}\right)\left(4e^{-2x} - e^{-4x}\right)$

$= 16e^{-4x} - 8e^{-6x} + e^{-8x}$

$\pi \int_0^{\ln 2} 16e^{-4x} - 8e^{-6x} + e^{-8x} \, dx$

$= \pi \left[-4e^{-4x} + \frac{4}{3}e^{-6x} - \frac{1}{8}e^{-8x}\right]_0^{\ln 2}$

$= \pi \left[\left(-4e^{-4\ln 2} + \frac{4}{3}e^{-6\ln 2} - \frac{1}{8}e^{-8\ln 2}\right) - \left(-4 + \frac{4}{3} - \frac{1}{8}\right)\right]$

$= \pi \left[-4(2^{-4}) + \frac{4}{3}(2^{-6}) - \frac{1}{8}(2^{-8}) + \frac{67}{24}\right]$

$= \pi \left[-\frac{1441}{6144} + \frac{67}{24}\right] = \frac{5247}{2048}\pi$