June 2010 Core 3 Worked solutions.

① Let $f(x) = 3^x - 10 + x^3$

(a)

$f(1) = 3 - 10 + 1 = -6$

$f(2) = 9 - 10 + 8 = +7$

Change of sign indicates that solution to $f(x)=0$ lies between 1 and 2.

(b)(i) $3^x = 10 - x^3$

$x^3 = 10 - 3^x$

$x = \sqrt[3]{10-3^x}$

(ii) $x_1 = 1$

$x_2 = \sqrt[3]{10-3} = \sqrt[3]{7} = 1.913$

$x_3 = \sqrt[3]{10-3^{\sqrt[3]{7}}} = 1.221$

② (a) (i) $y = \sec(0) = \frac{1}{\cos(0)} = 1$

(ii) see diagram over page.

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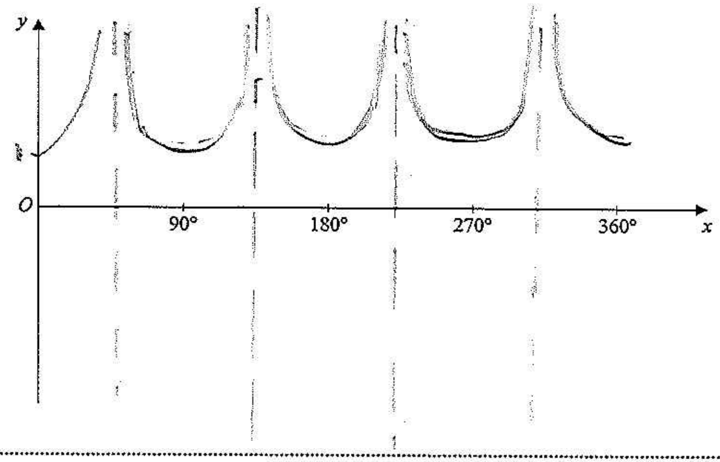
QUESTION PART REFERENCE

(a)(ii)



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3 (a) Find $\dfrac{dy}{dx}$ when:

(i) $y = \ln(5x - 2)$; *(2 marks)*

(ii) $y = \sin 2x$. *(2 marks)*

(b) The functions f and g are defined with their respective domains by

$\mathrm{f}(x) = \ln(5x - 2)$, for real values of x such that $x \geqslant \frac{1}{2}$

$\mathrm{g}(x) = \sin 2x$, for real values of x in the interval $-\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{4}$

(i) Find the range of f. *(2 marks)*

(ii) Find an expression for $\mathrm{gf}(x)$. *(1 mark)*

(iii) Solve the equation $\mathrm{gf}(x) = 0$. *(3 marks)*

(iv) The inverse of g is g^{-1}. Find $\mathrm{g}^{-1}(x)$. *(2 marks)*

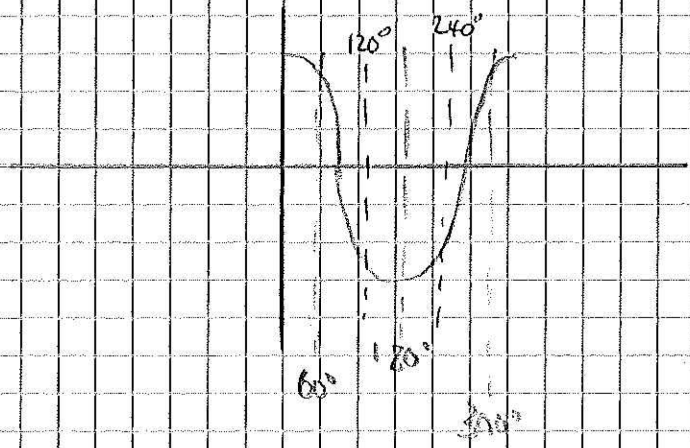
QUESTION PART REFERENCE

(b) $\sec x = 2$

$\frac{1}{\cos x} = 2$ $\cos x = \frac{1}{2}$

$x = 60^\circ$

300°



(c) $|\sec(2x-10)| = 2$

$\frac{1}{\cos(2x-10)} = 2$ $\frac{1}{\cos(2x-10)} = -2$

$\cos(2x-10) = \frac{1}{2}$ $\cos(2x-10) = -\frac{1}{2}$

$2x-10 = 60^\circ$ $2x - 10 = 120^\circ$

300° 240°

$x = 35^\circ$ $x = 65^\circ$

$x = 155^\circ$ $x = 125^\circ$

3 (a)

(i) $\frac{dy}{dx} = 5 \times \frac{1}{5x-2} = \frac{5}{5x-2}$

(ii) $\frac{dy}{dx} = 2 \times \cos 2x = 2\cos 2x$

(b) $f(x) = \ln(5x-2) \qquad x \geq \frac{1}{2}$

$g(x) = \sin 2x \qquad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

(i) $f(\frac{1}{2}) = \ln(\frac{5}{2} - 2) = \ln(\frac{1}{2})$

$f(x) \geq \ln \frac{1}{2}$

(ii) $gf(x)) = \sin[2(\ln(5x-2))]$

(iii) $0 = \sin[2\ln(5x-2)]$

$2\ln(5x-2) = 0$

$5x - 2 = 1$

$5x = 3 \Rightarrow x = \frac{3}{5}$

(iv) $g(x) = \sin 2x$

$y = \sin 2x$

$g^{-1}(x): \quad x = \sin 2y$

$y = \frac{1}{2}\sin^{-1}(x)$

$g^{-1}(x) = \frac{1}{2}\sin^{-1}(x)$

4(a) $h = \dfrac{2 - 0.5}{6} = \dfrac{\frac{3}{2}}{6} = \dfrac{1}{4}$

$x_0 = 0.5 \quad x_1 = 0.75 \quad x_2 = 1 \quad x_3 = 1.25 \quad x_4 = 1.5 \quad x_5 = 1.75 \quad x_6 = 2$

$y_0 = \frac{4}{9} \quad y_1 = \frac{48}{91} \quad y_2 = \frac{1}{2} \quad y_3 = \frac{80}{189} \quad y_4 = \frac{12}{35} \quad y_5 = \frac{112}{407} \quad y_6 = \frac{2}{9}$

$$\int \approx \frac{1}{3} \times \frac{1}{4} \left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$= 0.605$$

(b) $\displaystyle\int \frac{x^2}{1+x^3}\,dx$

$$\frac{d}{dx} \ln(1+x^3) = \frac{1}{1+x^3} \times 3x^2$$

$$\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3} \int \frac{3x^2}{1+x^3}\,dx = \frac{1}{3}\ln(1+x^3) + C$$

$$\int_0^1 \frac{x^2}{1+x^3} = \left[\frac{1}{3}\ln(1+x^3)\right]_0^1 = \left[\frac{1}{3}\ln 2\right] = \frac{1}{9}\ln 2$$

5(a) Use $\cot^2 x + 1 = \operatorname{cosec}^2 x$

$$10(1 + \cot^2 x) = 16 - 11\cot x$$

$$10\cot^2 x = 6 - 11\cot x \Rightarrow 10\cot^2 x + 11\cot x - 6 = 0$$

(b) let $X = \cot x$ then $10X^2 + 11X - 6 = 0$

$$(5X - 2)(2X + 3) = 0$$

$$X = \frac{2}{5} \qquad X = -\frac{3}{2}$$

so: $\tan x = \frac{5}{2} \qquad \tan x = -\frac{2}{3}$

6 (a) $y = 0 \Rightarrow \ln x = 0$

$x = 1$

$A = (1, 0)$

(b) let $f(x) = \ln x$ $\quad g(x) = x$ $\quad$ then $y = \dfrac{f(x)}{g(x)}$

$f'(x) = \dfrac{1}{x}$ $\quad g'(x) = 1$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{1 - \ln x}{x^2}$$

$$= \frac{1}{x^2} - \frac{\ln x}{x^2}$$

$\dfrac{dy}{dx} = 0$ at stationary point:

$$\frac{1 - \ln x}{x^2} = 0 \Rightarrow \ln x = 1$$

$$x = e$$

$$y = \frac{\ln e}{e} = e^{-1}$$

$$B = (e, e^{-1})$$

(c) when $x = e^3$

$$\frac{dy}{dx} = \frac{1 - \ln e^3}{e^6} = \frac{-2}{e^6}$$

gradient of normal $= \dfrac{e^6}{2}$

7 (a)(i) Let $u = x \quad v' = \cos 4x$

$u' = 1 \quad v = \frac{1}{4}\sin 4x$

$\int x\cos 4x\,dx = uv - \int u'v = \frac{1}{4}x\sin 4x - \int \frac{1}{4}\sin 4x$

$= \frac{1}{4}x\sin 4x + \frac{1}{4}\times\frac{1}{4}\cos 4x + C$

$= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C$

(ii) Let $u = x^2 \quad v' = \sin 4x$

$u' = 2x \quad v = -\frac{1}{4}\cos 4x$

$\int x^2\sin 4x\,dx = -\frac{1}{4}x^2\cos 4x + \int \frac{2x}{4}\cos 4x$

$= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x$

from (a) $\int x\cos 4x = \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C$

$\Rightarrow \int x^2\sin 4x\,dx = -\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C$

(b) $V = \pi\int_0^{0.2} y^2\,dx = \pi\int_0^{0.2} 64x^2\sin 4x$

$= 64\pi\left[-\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x\right]_0^{0.2}$

$= 64\pi\left[\left(-\frac{1}{4}(0.2)^2\cos(0.8) + \frac{1}{8}(0.2)\sin(0.8) + \frac{1}{32}\cos(0.8)\right) - \frac{1}{32}\right]$

$= 0.299365$

$= 0.299$ (3sf)

⑧ (a) Translation by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and a stretch with scale factor $\frac{1}{2}$ in x-direction.

(b) $x = 0 \qquad y = 4 + 2 = 6$

$(0, 6)$

(c) (i) Intersection of the 2 curves.

$$4e^{-2x} + 2 = e^{2x} - 1$$

$(\times e^{2x})$

$$4 + 2e^{2x} = e^{4x} - e^{2x}$$

$$e^{4x} - 3e^{2x} - 4 = 0$$

$$(e^{2x})^2 - 3e^{2x} - 4 = 0$$

(ii) Let $X = e^{2x}$

$$X^2 - 3X - 4 = 0$$

$$(X - 4)(X + 1)$$

$$X = 4 \qquad X = -1$$

$$e^{2x} = 4 \qquad e^{2x} = -1$$

not possible.

$$2x = \ln 4$$

$$x = \frac{1}{2}\ln 4$$

$$= \ln 4^{\frac{1}{2}} = \ln 2$$

$$x = \ln 2$$

(d) $\int_0^{\ln 2} 4e^{-2x} + 2 \; - \int_0^{\ln 2} e^{2x} - 1 \quad dx$

$= \int_0^{\ln 2} 4e^{-2x} + 2 - e^{2x} + 1 \quad dx$

$= \int_0^{\ln 2} 4e^{-2x} - e^{2x} + 3 \quad dx$

$= \left[-2e^{-2x} - \frac{1}{2}e^{2x} + 3x \right]_0^{\ln 2}$

$= \left[-2e^{-2\ln 2} - \frac{1}{2}e^{2\ln 2} + 3\ln 2 \right] - \left[-2 - \frac{1}{2} \right]$

$= \left[-2e^{\ln 2^{-2}} - \frac{1}{2}e^{\ln 4} + 3\ln 2 \right] + \frac{5}{2}$

$= (-2)(2^{-2}) - \frac{1}{2}(4) + 3\ln 2 + \frac{5}{2}$

$= 3\ln 2$