January 2010 Core 3 Worked Solutions

① (a) let $f(x) = e^{-4x}$ $\quad g(x) = x^2 + 2x - 2$

then $y = f(x)\,g(x)$

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

$$f'(x) = -4e^{-4x} \qquad g'(x) = 2x + 2$$

$$\frac{dy}{dx} = -4e^{-4x}(x^2+2x-2) + e^{-4x}(2x+2)$$

$$= -4e^{-4x}(x^2+2x-2) + 2e^{-4x}(x+1)$$

$$= 2e^{-4x}\left(-2(x^2+2x-2) + x + 1\right)$$

$$= 2e^{-4x}(-2x^2 - 3x + 5) = 2e^{-4x}(5-3x-2x^2)$$

(b) $\frac{dy}{dx} = 0$ at stationary points

$$2e^{-4x}(5-3x-2x^2) = 0$$

$$2x^2 + 3x - 5 = 0$$

$$(2x+5)(x-1) = 0$$

$$x = \frac{-5}{2} \qquad x = 1$$

$$y = e^{+4\left(\frac{5}{2}\right)}\left(\left(\tfrac{5}{2}\right)^2 - 2\left(\tfrac{5}{2}\right) - 2\right) \qquad y = e^{-4}(1+2-2)$$

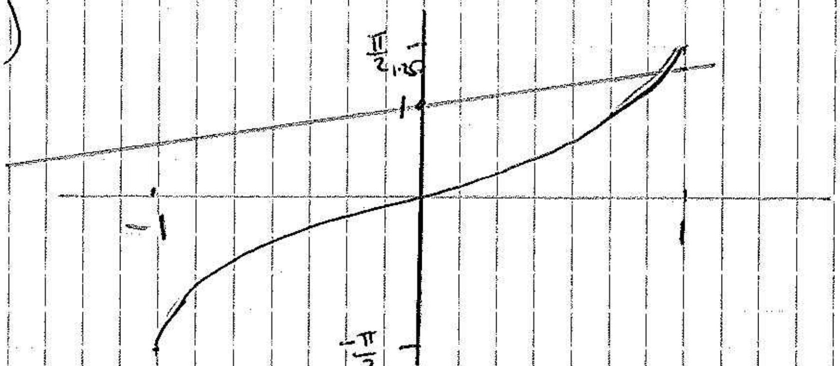
$$= e^{10}\left(\frac{25}{4} - \frac{20}{4} - \frac{8}{4}\right) \qquad = e^{-4}$$

$$= \frac{-3}{4}e^{10}$$

$$\left(-\frac{5}{2}, \frac{-3}{4}e^{10}\right) \qquad \left(1, e^{-4}\right)$$

2(a)

(i)



$A = \left(1, \frac{\pi}{2}\right)$ $B = \left(-1, \frac{-\pi}{2}\right)$

(ii) $y = \frac{1}{4}x + 1$

gradient $= \frac{1}{4} = \frac{0.25}{1}$ $\left(\frac{\Delta y}{\Delta x}\right)$ across 1, up 0.25

The curve and line have only one point of intersection. $\Rightarrow$ one solution.

(b) Let $f(x) = \sin^{-1}(x) - \frac{1}{4}x - 1$

$f(0.5) = \sin^{-1}(0.5) - \frac{1}{4}\left(\frac{1}{2}\right) - 1 = -0.6$

$f(1) = \sin^{-1}(1) - \frac{1}{4} - 1 = 0.3$

Change of sign indicates that $f(x) = 0$ has solution: $0.5 < x < 1$.

So $0.5 < \alpha < 1$

(c)(i) $x_1 = 0.5$

$x_2 = \sin\left(\frac{1}{4}(0.5) + 1\right) = 0.902$

$x_3 = \sin\left(\frac{1}{4}x_2 + 1\right) = 0.941$

(substitute in value of x_2 before it was rounded to 3dp)

(ii) See diagram.

(3) (a) $\operatorname{cosec} x = 3$

$\frac{1}{\sin x} = 3 \Rightarrow \sin x = \frac{1}{3}$

$x = 0.34$

2.80

(b) $1 + \cot^2 x = \operatorname{cosec}^2 x$

$\cot^2 x = \operatorname{cosec}^2 x - 1$

$\operatorname{cosec}^2 x - 1 = 11 - \operatorname{cosec} x$

$\operatorname{cosec}^2 x + \operatorname{cosec} x - 12 = 0$

$(\operatorname{cosec} x + 4)(\operatorname{cosec} x - 3) = 0$

$\operatorname{cosec} x = -4$ $\qquad$ $\operatorname{cosec} x = 3$

$\sin x = \frac{-1}{4}$ $\qquad$ $\Rightarrow x = 0.34$

2.80

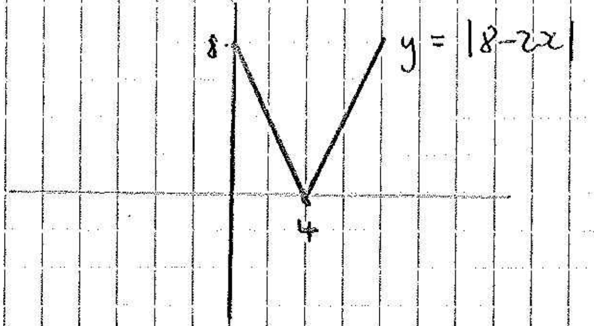
$x = -0.25$

3.39

6.03

$\boxed{x = 0.34, 2.80, 3.39, 6.03}$

(4) (a)



(b)

$$|8-2x| = 4$$

$8 - 2x = 4$

$2x = 4$

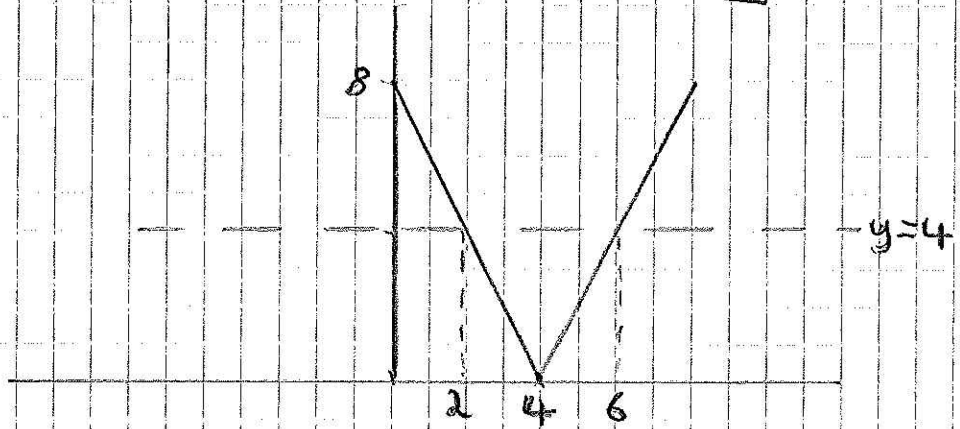
$\underline{\underline{x = 2}}$

$-8 + 2x = 4$

$2x = 12$

$\underline{\underline{x = 6}}$

(c)



From sketch above you can see that

$|8-2x| > 4$ when

$x < 2, \quad x > 6$

⑤ (a) $h = \frac{12-0}{4} = 3$ $\qquad y = \ln(x^2+5)$

$x_0 = 0 \quad x_1 = 3 \quad x_2 = 6 \quad x_3 = 9 \quad x_4 = 12$

$x_{\frac{1}{2}} = 1.5 \quad x_{\frac{3}{2}} = 4.5 \quad x_{\frac{5}{2}} = 7.5 \quad x_{\frac{7}{2}} = 10.5$

$y_{\frac{1}{2}} = 1.9810 \quad y_{\frac{3}{2}} = 3.2289 \quad x_{\frac{5}{2}} = 4.1150 \quad x_{\frac{7}{2}} = 4.7471$

$\int \approx 3 \times \left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + y_{\frac{5}{2}} + y_{\frac{7}{2}}\right)$

$= 42.216$

$= 42.2$ (3sf)

(b)(i) $y = \ln(x^2+5)$

$e^y = e^{\ln(x^2+5)}$

$e^y = x^2 + 5$

$x^2 = e^y - 5$

(ii) $V = \pi \int_5^{10} x^2 \, dy$

$= \pi \int_5^{10} e^y - 5 \quad dy$

$= \pi \left[e^y - 5y\right]_5^{10} = \pi \left[e^{10} - 50 - e^5 + 25\right]$

$= \pi \left(e^{10} - e^5 - 25\right)$

5(c) Stretch: $y = \ln\left[\left(\frac{x}{4}\right)^2 + 5\right]$

Translation: $y = \ln\left[\left(\frac{x}{4}\right)^2 + 5\right] + 3$

$$f(x) = \ln\left[\left(\frac{x}{4}\right)^2 + 5\right] + 3$$

6(a) $e^{2x} > 0$

$e^{2x} - 3 > 0 - 3$

$\boxed{f(x) > -3}$

(b)(i) $f(x)$: $y = e^{2x} - 3$

$f^{-1}(x)$: $x = e^{2y} - 3$

$e^{2y} = x + 3$

$2y = \ln(x+3)$

$y = \frac{1}{2}\ln(x+3)$

$f^{-1}(x) = \frac{1}{2}\ln(x+3)$

(ii) $\frac{1}{2}\ln(x+3) = 0$

$\ln(x+3)^{\frac{1}{2}} = 0$

$(x+3)^{\frac{1}{2}} = 1$

$x + 3 = 1$

$x = -2$

(6) (i) $gf(x) = g(f(x))$

$$= \frac{1}{3(e^{2x}-3)+4}$$

$$= \frac{1}{3e^{2x}-5}$$

(ii) $1 = \dfrac{1}{3e^{2x}-5} \Rightarrow 3e^{2x}-5 = 1$

$$3e^{2x} = 6$$

$$e^{2x} = 2$$

$$2x = \ln 2$$

$$x = \tfrac{1}{2}\ln 2$$

(7) $y = \tan 4x = \dfrac{\sin 4x}{\cos 4x}$

(a) let $f(x) = \sin 4x \qquad g(x) = \cos 4x$

$f'(x) = 4\cos 4x \qquad g'(x) = -4\sin 4x$

$$\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{4\cos^2 4x + 4\sin^2 4x}{\cos^2 4x}$$

$$= \frac{4(\cos^2 4x + \sin^2 4x)}{\cos^2 4x} = \frac{4}{\cos^2 4x} = 4\sec^2 4x$$

$\sec^2 A = 1 + \tan^2 A$

So $\dfrac{dy}{dx} = 4(1+\tan^2 4x)$

$\underline{\underline{p = 4}}$

(b) $\frac{dy}{dx} = 4 + 4\tan^2 4x$

$\frac{d}{dx}\tan^2 4x \qquad$ let $f(x) = \tan 4x \qquad g(x) = \tan 4x$

then product rule becomes: $2f'(x)f(x)$

$$\frac{d}{dx}(\tan^2 4x) = 2\left(4+4\tan^2 4x\right)\cdot\tan 4x$$
$$= 8\tan 4x + 8\tan^3 4x$$

so
$$\frac{d^2y}{dx^2} = 4\left(8\tan 4x + 8\tan^3 4x\right)$$
$$= 32\tan 4x\left(1+\tan^2 4x\right)$$
$$= 32\tan 4x\sec^2 4x$$
$$= 32y\left(1+y^2\right)$$

8 (a) let $u = x \qquad v' = \sin(2x-1)$

$u' = 1 \qquad v = -\frac{1}{2}\cos(2x-1) \qquad uv - \int u'v$

$$\int x\sin(2x-1)\,dx = -\frac{1}{2}x\cos(2x-1) + \frac{1}{2}\int\cos(2x-1)\,dx$$
$$= -\frac{1}{2}x\cos(2x-1) + \frac{1}{2}\left[\frac{1}{2}\sin(2x-1)\right] + C$$
$$= \frac{1}{4}\sin(2x-1) - \frac{1}{2}x\cos(2x-1) + C$$

8(b) $u = 2x - 1$ $\qquad x = \dfrac{u+1}{2}$

$\dfrac{du}{dx} = 2 \qquad \dfrac{dx}{du} = \dfrac{1}{2}$

$$\int \frac{x^2}{2x-1}\,dx = \int \frac{\left(\frac{u+1}{2}\right)^2}{u} \times \frac{1}{2}\,du$$

$$= \frac{1}{2}\int \frac{1}{4u}\left(u^2 + 2u + 1\right)\,du$$

$$= \frac{1}{8}\int u + 2 + u^{-1}\,du$$

$$= \frac{1}{8}\left[\frac{1}{2}u^2 + 2u + \ln u\right] + C$$

$u = 2x - 1 \qquad u^2 = 4x^2 - 4x + 1$

Mark scheme stops here

$$\frac{1}{8}\left[\frac{1}{2}\left(4x^2 - 4x + 1\right) + 2(2x-1) + \ln(2x-1)\right] + C$$

$$= \frac{1}{8}\left[2x^2 - 2x + \frac{1}{2} + 4x - 2 + \ln(2x-1)\right] + C$$

$$= \frac{1}{8}\left[2x^2 + 2x - \frac{3}{2} + \ln(2x-1)\right] + C$$

$$= \frac{1}{16}\left(4x^2 + 4x - 3\right) + \frac{1}{8}\ln(2x-1) + C$$

Lots of equivalent ways to write this.