Core 3 - June 2009

(1) a) i) $\dfrac{\cos(x)}{2x+1} = \frac{1}{2} \rightarrow \dfrac{\cos(x)}{2x+1} - \frac{1}{2} = 0$

Let $f(x) = \dfrac{\cos(x)}{2x+1} - \frac{1}{2}$

$f(0) = \dfrac{\cos(0)}{2(0)+1} - \frac{1}{2} = \dfrac{1}{1} - \frac{1}{2} = \frac{1}{2}$

$f(\pi/2) = \dfrac{\cos(\pi/2)}{2(\pi/2)+1} - \frac{1}{2} = 0 - \frac{1}{2} = -\frac{1}{2}$

Sign change, therefore root lies between 0 and $\pi/2$

ii) $\dfrac{\cos(x)}{2x+1} = \dfrac{1}{2} \rightarrow \cos(x) = \dfrac{2x+1}{2}$

$\rightarrow 2\cos(x) = 2x+1$

$\rightarrow 2\cos(x) - 1 = 2x$

$\rightarrow x = \cos(x) - \frac{1}{2}$

iii) $x_1 = 0$

$x_2 = \cos(0) - \frac{1}{2} = \frac{1}{2}$

$x_3 = \cos(\frac{1}{2}) - \frac{1}{2} = 0.378$ (3dp)

b) i) $y = \dfrac{\cos(x)}{2x+1}$ $\quad u = \cos(x)$ $\quad v = 2x+1$

$du/dx = -\sin(x)$ $\quad dv/dx = 2$

$dy/dx = \dfrac{(2x+1)-\sin(x) - 2\cos(x)}{(2x+1)^2}$

$= \dfrac{-(2x+1)\sin(x) - 2\cos(x)}{(2x+1)^2}$

ii) when $x = 0$, $dy/dx = \dfrac{-(1)\sin(0) - 2\cos(0)}{(1)^2} = -2$

$\therefore$ gradient of normal $= \frac{1}{2}$

(2) a) $f(x)$ cannot be negative, so $f(x) \geq 0$

b) i) Let $y = \sqrt{2x+5}$

$\Rightarrow \quad x = \sqrt{2y+5}$

$x^2 = 2y + 5$

$x^2 - 5 = 2y$

$\Rightarrow \quad y = \dfrac{x^2-5}{2} = f^{-1}(x)$

ii) Same as Range of $f(x)$ = $x \geq 0$

c) i) $fg(x) = f\left(\dfrac{1}{4x+1}\right) = \sqrt{\left(\dfrac{2}{4x+1}\right) + 5}$

ii) $\sqrt{\dfrac{2}{4x+1} + 5} = 3$

$\dfrac{2}{4x+1} + 5 = 9$

$\dfrac{2}{4x+1} = 4$

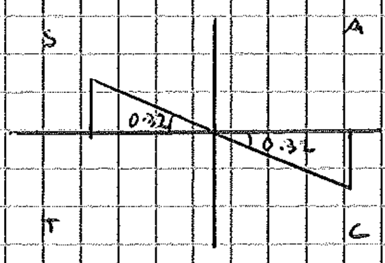
$2 = 16x + 4$

$16x = -2$

$x = -1/8$

(3) a) $\tan(x) = -1/3$

$x = -0.3217...$



$x = 2.82, \quad x = 5.96$

b) $3\sec^2(x) = 5(\tan(x) + 1)$ $\quad \sec^2(x) = \tan^2(x) + 1$

$\Rightarrow \quad 3(\tan^2(x) + 1) = 5\tan(x) + 5$

$3\tan^2(x) + 3 = 5\tan(x) + 5$

$\Rightarrow \quad 3\tan^2(x) - 5\tan(x) - 2 = 0$

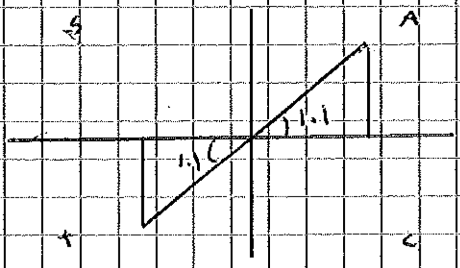
c) $3\tan^2(x) - 5\tan(x) - 2 = 0$

$(3\tan(x) + 1)(\tan(x) - 2) = 0$

$\tan(x) = -1/3$ $\qquad$ $\tan(x) = 2$

$x = 1.107$

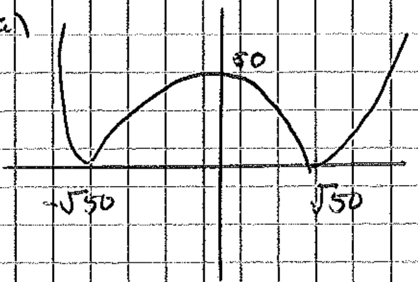
$x = 2.82, 5.96$
(from a)



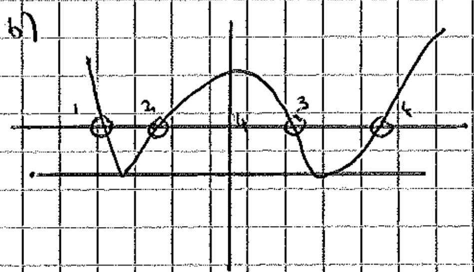
$x = 1.11, 4.25$

$x = 1.11, 2.82, 4.25, 5.96$

(4) a)



b)



For (1) & (4)

$-(50 - x^2) = 14$

$x^2 - 50 = 14$

$x^2 = 36$

$x = \pm 6$

For (2) & (3)

$50 - x^2 = 14$

$x^2 = 64$

$x = \pm 8$

Solutions, $x = -6, -8, 6, 8$

c) From graph, $x < -8$, $\quad -6 < x < 6$, $\quad x > 8$

d) $-x^2$ Reflection in x-axis

50 Translation $\begin{pmatrix} 0 \\ 50 \end{pmatrix}$

(5) a) $2\ln(x) = 5 \rightarrow \ln(x) = 5/2$

$\rightarrow x = e^{5/2}$

b) $2\ln(x) + \dfrac{15}{\ln(x)} = 11$

$\rightarrow 2[\ln(x)]^2 + 15 = 11\ln(x)$

$2[\ln(x)]^2 - 11\ln(x) + 15 = 0$

$\rightarrow (2\ln(x) - 5)(\ln(x) - 3) = 0$

$\downarrow \qquad\qquad \downarrow$

$\ln(x) = 5/2 \qquad\qquad \ln(x) = 3$

$\rightarrow x = e^{5/2} \qquad\qquad \rightarrow x = e^3$

(6) a) $y = \sqrt{100 - 4x^2}$

$y^2 = 100 - 4x^2$

$4x^2 = 100 - y^2$

$x^2 = 1/4\,(100 - y^2)$

$\text{Volume} = \pi \int x^2\, dy$

$= 1/4\,\pi \int_0^{10} 100 - y^2\, dy$

$= 1/4\,\pi \left[100y - y^3/3\right]_0^{10}$

$= 1/4\,\pi \left[100(10) - \dfrac{10^3}{3} - 0\right]$

$= 1/4\,\pi\,(2000/3) = 500\pi/3$

b)

x	y
0.5	9.950
1.5	9.539
2.5	8.660
3.5	7.141
4.5	4.359

$h = 1$

$\text{Area} = 1 \times (9.950 + 9.539 + 8.660 + 7.141 + 4.359)$

$= 39.6$ (3sf)

c) i) $y = (100 - 4x^2)^{1/2}$

$dy/dx = \frac{1}{2}(-8x)(100 - 4x^2)^{-1/2}$ {quick chain rule

when $x = 3$, $dy/dx = \frac{1}{2}(-24)(100 - 4(9))^{-1/2}$

$= -3/2$

ii) $x = 3$

$y = 8$

$m = -3/2$

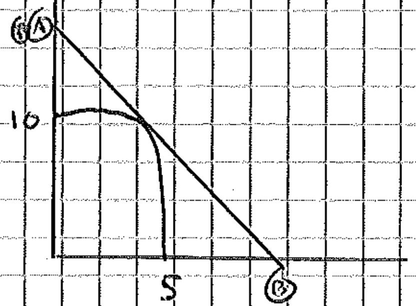
$y - 8 = -3/2(x - 3)$

$2y + 16 = -3(x - 3)$

$2y - 16 = -3x + 9$

$2y + 3x = 25$

d)



(A) $x = 0 \Rightarrow 2y = 25$
$\Rightarrow y = 12.5$

(B) $y = 0 \Rightarrow 3x = 25$
$\Rightarrow x = 25/3$

Area of $\Delta = \frac{1}{2} b \times h$

$= \frac{1}{2} \times 25/3 \times 12.5$

$= 625/12$

$\therefore$ shaded area $= 625/12 - 39.6 = 12.5$ (3sf)

7) a) $\int (t-1)\ln(t)$

$u = \ln(t)$ $\quad dv/dt = t - 1$

$du/dt = 1/t$ $\quad v = t^2/2 - t$

$= uv - \int v \, du/dt$

$= \ln(t)[t^2/2 - t] - \int 1/t\,(t^2/2 - t)$

$= \ln(t)[t^2/2 - t] - \int t/2 - 1$

$= \ln(t)[t^2/2 - t] - [t^2/4 - t]$

$= \ln(t)[t^2/2 - t] - t^2/4 + t + c$

b) $\int 4x \ln(2x+1)\, dx$

$t = 2x + 1$

$dt/dx = 2$

$dx = dt/2$

$\int 4x \ln(t)\, dt/2$

$t = 2x + 1$

$x = \frac{1}{2}(t - 1)$

$4x = 2t - 2$

$\int (2t-2)(\ln t)\, dt/2$

$= \int (t-1) \ln(t)\, dt$

c) Limits: $t = 2x + 1$

$1 \to 2(1) + 1 = 3$

$0 \to 2(0) + 1 = 1$

$$\int_1^3 (t-1)\ln(t) = \left[\ln(t)\left[t^2/2 - t\right] - t^2/4 + t\right]_1^3$$

$$= \left[\ln(3)\left[9/2 - 3\right] - 9/4 + 3\right] - \left[\ln(1)\left[1/2 - 1\right] - 1/4 + 1\right]$$

$$= 3/2 \ln(3) + 3/4 - 3/4$$

$$= 3/2 \ln(3)$$