Core 3 - May / June 2008

(1) a) $y = (3x+1)^5 \rightarrow dy/dx = 5 \times 3(3x+1)^4$

$= 15(3x+1)^4$

b) $y = \ln(3x+1) \rightarrow y = \ln(t)$ $\quad t = 3x+1$

$dy/dt = 1/t$ $\quad dt/dx = 3$

$\therefore dy/dx = 3/t = \dfrac{3}{3x+1}$

c) $y = (3x+1)^5 \ln(3x+1)$ $\quad u = (3x+1)^5$ $\quad v = \ln(3x+1)$

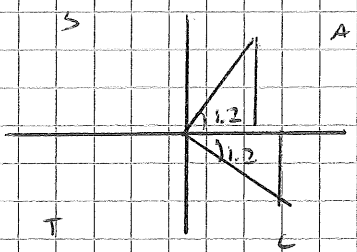
PRODUCT RULE $\quad du/dx = 15(3x+1)^4$ $\quad dv/dx = 3/3x+1$

$dy/dx = 15(3x+1)^4 \ln(3x+1) + \dfrac{3(3x+1)^5}{3x+1}$

$= 15(3x+1)^4 \ln(3x+1) + 3(3x+1)^4$

(2) a) $\sec(x) = 3 \rightarrow \cos(x) = 1/3$

$x = 1.2309...$



$x = 1.23, 5.05$

b) $\tan^2(x) = 2\sec(x) + 2$ $\quad \sec^2(x) = 1 + \tan^2(x)$

$\sec^2(x) - 1 = 2\sec(x) + 2$

$\rightarrow \sec^2(x) - 2\sec(x) - 3 = 0$

c) $(\sec(x) - 3)(\sec(x) + 1) = 0$

$\sec(x) = 3$

$x = 1.23, 5.05$

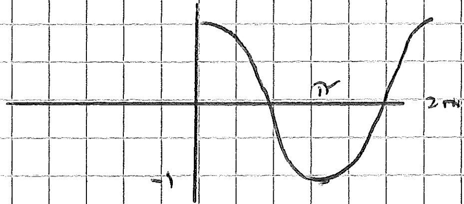
(from a)

$\sec(x) = -1$

$\rightarrow \cos(x) = -1$

$x = 3.14 \ (\pi)$

(from graph)



$x = 1.23, 3.14, 5.05$

(3) a) $y = x\cos(2x)$ $\qquad u = x \qquad v = \cos(2x)$

$\qquad du/dx = 1 \qquad dv/dx = -2\sin(2x)$

$dy/dx = \cos(2x) - 2x\sin(2x)$

b) i) At A, $dy/dx = 0$

$\therefore \cos(2x) - 2x\sin(2x) = 0$

$\div$ by $\cos(2x)$ $\Big\}$ $1 - 2x\tan(2x) = 0$

at A, $x = \alpha$ ~~$2x\tan(2x)$~~

$\rightarrow 1 - 2\alpha\tan(2\alpha) = 0$

ii) ~~2α~~ Let $f(\alpha) = 1 - 2\alpha\tan(2\alpha)$

$f(0.4) = 1 - 2(0.4)\tan(0.8) = 0.176\ldots$

$f(0.5) = 1 - 2(0.5)\tan(1) = -0.557\ldots$

change of sign, $\therefore$ root lies between 0.4 & 0.5

iii) $1 - 2\alpha\tan(2\alpha) = 0$

$1 = 2\alpha\tan(2\alpha)$

$1/2\alpha = \cancel{2}\tan(2\alpha)$

$\tan^{-1}(1/2\alpha) = 2\alpha$

$\rightarrow \alpha = 1/2\tan^{-1}(1/2\alpha)$ $\qquad$ $\alpha = x!$

iv) $x_1 = 0.4$

$x_2 = 1/2\tan^{-1}(1/2\times0.4) = 0.448\ldots$

$x_3 = 1/2\tan^{-1}(1/2\times0.448) = 0.4200\ldots = 0.42$ (2sf)

c) $\int x\cos(2x)\,dx$

~~$u = x$~~ ~~$du/dx = 1$~~ $\qquad$ ~~$v = \cos(2x)$~~ ~~$dv/dx = -2\sin(2x)$~~

$$\boxed{\begin{array}{ll} u = x & dv/dx = \cos(2x) \\ du/dx = 1 & v = 1/2\sin(2x) \end{array}}$$

$uv - \int v\,du/dx$

$= 1/2x\sin(2x) - \int 1/2\sin(2x)$

$= \left[1/2\,x\sin(2x) + 1/4\cos(2x)\right]_0^{0.5}$

$= 1/2(0.5)\sin(1) + 1/4\cos(1) - 0 - 1/4\cos(0)$

$= 0.095443\ldots = 0.0954$ (3sf)

4) a) Range $f(x)$: ≥ 0

b) i) let $y = \frac{1}{2x-3}$

$y(2x-3) = 1$

$2x - 3 = 1/y$

$2x = \frac{1}{y} + 3$

$x = \frac{1}{2y} + 3/2$

$\Rightarrow y = \frac{1}{2x} + 3/2$

$\boxed{OR}$

$y(2x-3) = 1$

$2xy - 3y = 1$

$2xy = 1 + 3y$

$x = \frac{1 + 3y}{2y}$

$\Rightarrow y = \frac{1 + 3x}{2x}$

ii) Range $g^{-1}(x)$ = Domain $g(x)$ $\rightarrow$ $\neq 3/2$

c) $fg(x) = f\left(\frac{1}{2x-3}\right) = \left(\frac{1}{2x-3}\right)^2$

$\left(\frac{1}{2x-3}\right)^2 = 9$

$\frac{1}{2x-3} = \pm 3$

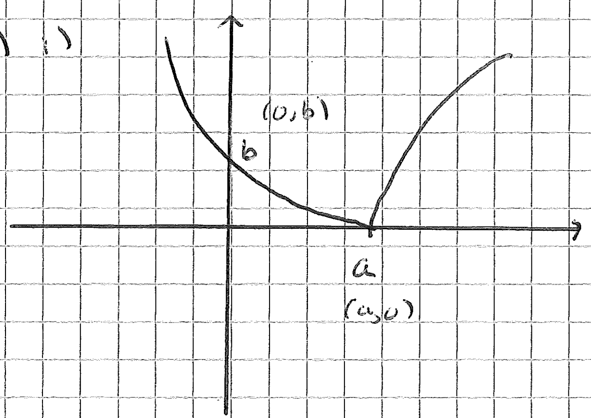
$2x - 3 = \pm 1/3$

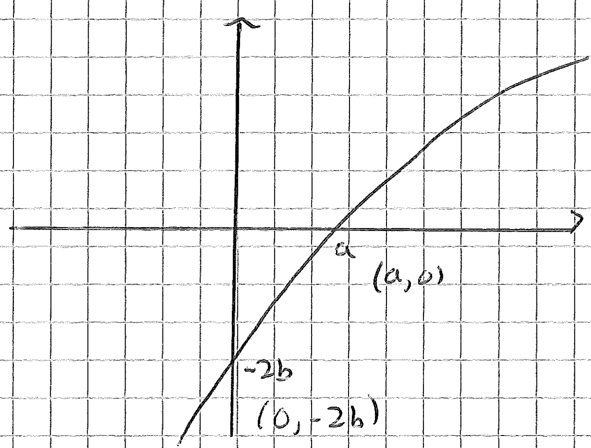
$2x = \pm 1/3 + 3$

$x = \frac{\pm 1/3 + 3}{2}$

$x = 5/3, 4/3$

5) a) i)





b) i) $\boxed{+1}$ Translation $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\boxed{4}$ Stretch, scale factor 4, parallel to y axis

$\boxed{-2}$ Translation $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

ii) crosses y axis when $x = 0$

$$y = 4\ln(1) - 2 = -2 \qquad (0, -2)$$

crosses x axis when $y = 0$

$$0 = 4\ln(x+1) - 2$$

$$2 = 4\ln(x+1)$$

$$1/2 = \ln(x+1)$$

$$e^{1/2} = x + 1$$

$$x = e^{1/2} - 1 \qquad (e^{1/2} - 1, 0)$$

6) a) $y = (e^{3x} + 1)^{1/2}$

$y = t^{1/2}$ $\qquad t = e^{3x} + 1$

$dy/dt = 1/2\, t^{-1/2}$ $\qquad dt/dx = 3e^{3x}$

$$\therefore dy/dx = \frac{3e^{3x}}{2\sqrt{t}} = \frac{3e^{3x}}{2\sqrt{e^{3x}+1}}$$

when $x = \ln(2)$ $\quad dy/dx = \dfrac{3e^{3\ln(2)}}{2\sqrt{e^{3\ln(2)}+1}}$

$$= \frac{3 \times 8}{2 \times \sqrt{9}} = 4$$

b)

x	y
0.25	1.7655
0.75	3.2385
1.25	6.5971
1.75	13.841

$h = 0.5$

Area $= 0.5 \times 25.4421$

$= 12.7210$

$= 12.7$ (3sf)

c) Volume $= \pi \int y^2 \, dx$ $\qquad y = (e^{3x} + 1)^{1/2}$

$\therefore y^2 = e^{3x} + 1$

$$= \pi \int e^{3x} + 1 \, dx$$

$$= \pi \left[1/3\, e^{3x} + x \right]_0^2$$

$$= \pi \left[1/3\, e^6 + 2 - 1/3\, e^0 - 0 \right]$$

$$= \pi \left[1/3\, e^6 + 5/3 \right] = \pi/3 \left[e^6 + 5 \right]$$

7 a) $y = \dfrac{\sin\theta}{\cos\theta}$ $\quad u = \sin\theta \quad v = \cos\theta$

$du/d\theta = \cos\theta \quad dv/d\theta = -\sin\theta$

$$dy/d\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

b) $x = \sin(\theta)$

$x^2 = \sin^2(\theta)$ $\quad$ use $\sin^2\theta + \cos^2\theta = 1$

$x^2 = 1 - \cos^2(\theta)$

$\cos^2(\theta) = 1 - x^2$

$\rightarrow \cos(\theta) = \sqrt{1 - x^2}$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{x}{\sqrt{1 - x^2}}$$

c) $\displaystyle\int \frac{1}{(1 - x^2)^{3/2}}\,dx$ $\quad x = \sin(\theta)$

$dx/d\theta = \cos(\theta)$

$dx = \cos(\theta)\,d\theta$

$$= \int \frac{1}{(1 - \sin^2(\theta))^{3/2}} \cos\theta\,d\theta$$

$1 - \sin^2\theta = \cos^2\theta$

$$= \frac{1}{(\cos^2\theta)^{3/2}} \cos\theta\,d\theta$$

$$= \frac{1}{(\cos^2\theta)^3} \cos\theta\,d\theta$$

$$= \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta$$

$$= \tan(\theta)$$

$$= \frac{x}{\sqrt{1 - x^2}}$$