Core 3 - June 2007

(1) a) $y = \ln(x) \qquad dy/dx = 1/x$

b) $y = (x+1)\ln(x) \qquad u = x+1 \qquad v = \ln(x)$

$dy/dx = \ln(x) + 1/x\,(x+1) \qquad du/dx = 1 \qquad dv/dx = 1/x$ {Product Rule}

$= \ln(x) + 1 + 1/x$

c) $x = 1$

$y = 2\ln(x) = 0$

$dy/dx = \ln(x) + 1 + 1/x$

At $x=1$ gradient $= \ln(1) + 1 + 1 = 2 \rightarrow$ gradient of normal $= -1/2$

$y - y_1 = m(x - x_1)$

$y - 0 = -1/2\,(x-1) \Rightarrow y = -1/2\,(x-1)$

(2) a) $y = (x-1)^4 \qquad dy/dx = 4(x-1)^3$

b) $y^2 = \left[2\sqrt{(x-1)^3}\right]^2 = 4(x-1)^3$

Volume $= \pi \int_2^4 y^2 \, dx$

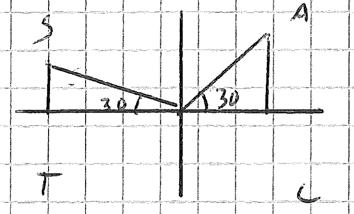
$= \pi \int_2^4 4(x-1)^3 \, dx = \pi\left[(x-1)^4\right]_2^4$

$= \pi\left[3^4 - 1^4\right] = 80\pi$

c) $(x-1)$ = Translation $\begin{pmatrix}1\\0\end{pmatrix}$

2 : stretch, SF 2, parallel to y axis

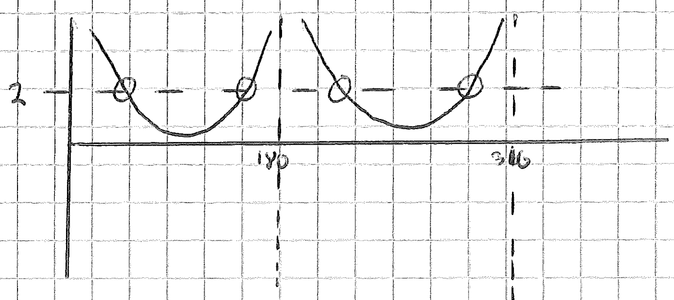
(3) a) $\operatorname{cosec}(x) = 2 \rightarrow \sin(x) = 1/2$



$x = 30°, 150°$

b) i) 1 $[90, 1]$

ii)



c) By symmetry:

$x = 30, 150$

$210, 330$

(4) a)

		y
x_0	1	3
x_1	1.25	3.9482..
x_2	1.5	5.1962..
x_3	1.75	6.8385
x_4	2	9

$h = 0.25$

$A = \frac{1}{3} \times 0.25 \times (3 + 9 + 4(3.948.. + 6.838) + 2(5.1962))$

$= 5.46$ (3sf)

b) i) At intersection: $3^x = x + 3 \rightarrow 3^x - x - 3 = 0$

Let $f(x) = 3^x - x - 3$

$f(0.5) = -1.7679...$

$f(1.5) = 0.69615..$

Sign change, ∴ root lies between 0.5 and 1.5

ii) $3^x = x + 3$

$x \ln(3) = \ln(x+3) \rightarrow x = \frac{\ln(x+3)}{\ln(3)}$

iii) $x_1 = 0.5$

$x_2 = \frac{\ln(0.5+3)}{\ln(3)} = 1.1403...$

$x_3 = \frac{\ln(1.140.. + 3)}{\ln(3)} = 1.2932.. = 1.3$ (2sf)

iv) See Insert

(5) a) $f(x) = \sqrt{x - 2}$ Range: $f(x) \geq 0$ cannot be negative!

b) i) $fg(x) = f(1/x) = \sqrt{1/x - 2}$

ii) $\sqrt{1/x - 2} = 1 \rightarrow 1/x - 2 = 1$

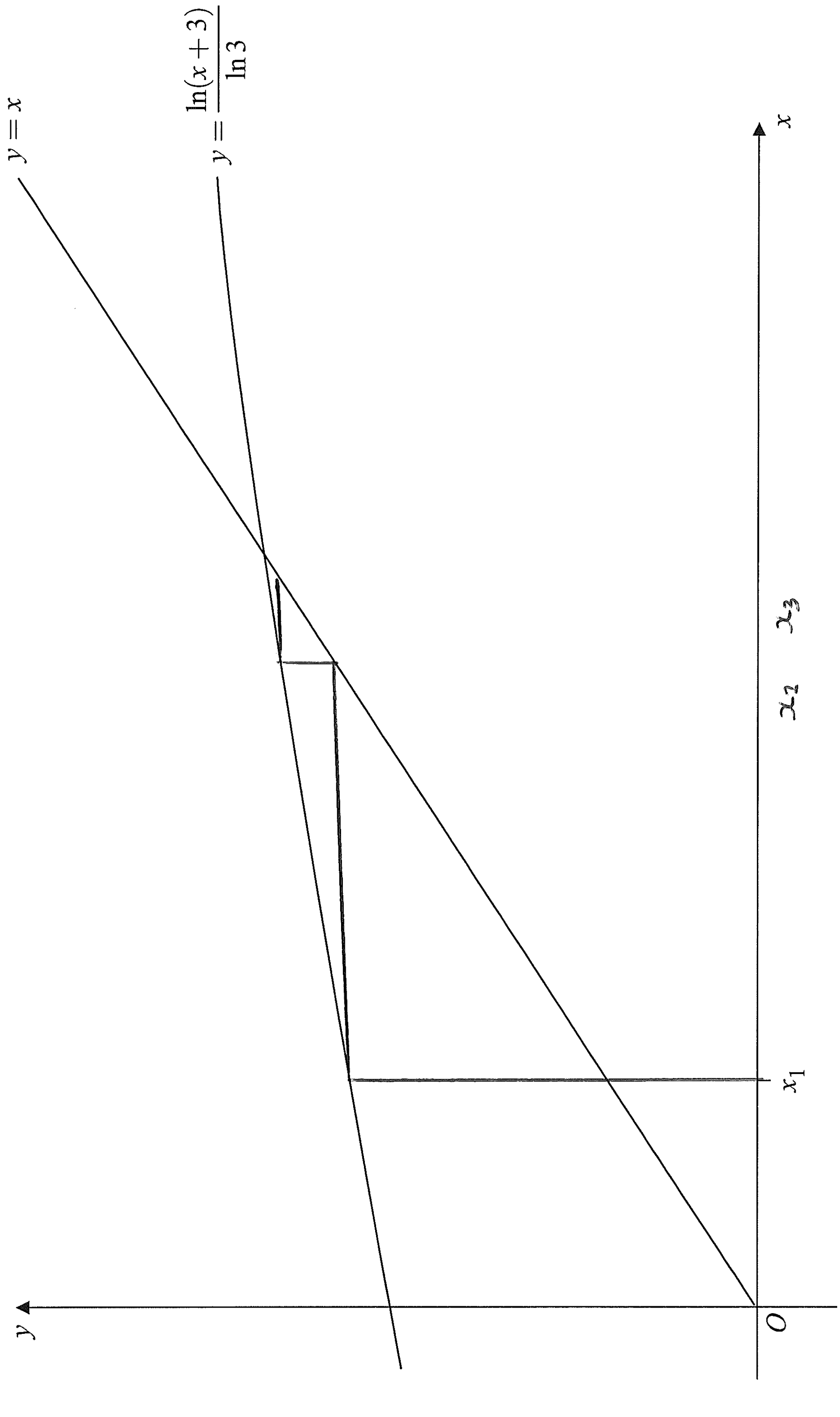
$1/x = 3 \rightarrow x = 1/3$

c) Let $y = \sqrt{x - 2}$

$y^2 = x - 2$

$x = y^2 + 2 \rightarrow y = x^2 + 2 = f^{-1}(x)$

Figure 1 (for use in Question 4)



(6) a) $\int x e^{5x}$

$u = x \qquad \frac{dv}{dx} = e^{5x}$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{5} e^{5x}$

$= uv - \int \frac{du}{dx} \, dv$

$= \frac{x}{5} e^{5x} - \int \frac{1}{5} e^{5x} \, dx$

$= \frac{x}{5} e^{5x} - \frac{1}{25} e^{5x} + c$

b) i) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})} \, dx$

$u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$= \int \frac{2\sqrt{x}}{\sqrt{x}(1+u)} \, du \qquad \to dx = 2\sqrt{x} \, du$

$= \int \frac{2}{1+u} \, du$

ii) change limits: $u = \sqrt{x} \qquad 1 \to 1, \quad 9 \to 3$

$\int_1^3 \frac{2}{1+u} = \left[2 \ln(1+u) \right]_1^3$

$= 2\ln(4) - 2\ln(2)$

$= 2\ln(2) = \ln(4)$

(7) a) $y = (x^2 - 3)e^x$

Product Rule

i) $\frac{dy}{dx} = 2xe^x + e^x(x^2-3)$

$u = x^2 - 3 \qquad v = e^x$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = e^x$

ii) $\frac{d^2y}{dx^2} = 2e^x + 2xe^x +$

$2xe^x + e^x(x^2-3)$

$\{ 2xe^x$

$u = 2x \qquad v = e^x$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = e^x$

b) i) At st point, $\frac{dy}{dx} = 0$

$2xe^x + e^x(x^2-3) = 0$

$e^x(2x + x^2 - 3) = 0$

$e^x(x^2 + 2x - 3) = 0$

$e^x(x+3)(x-1) = 0$

$\downarrow \qquad \downarrow x = -3 \qquad x = 1$

No solution

ii) $\boxed{x = 1}$

$\frac{d^2y}{dx^2} = 2e^1 + 2e^1 + 2e^1 + e(-2)$

$= 10.873$

$=$ MINIMUM

$\boxed{x = -3}$

$\frac{d^2y}{dx^2} = -0.2$

$=$ MAXIMUM

(8) a) $\int \sec^2(x)\,dx = \tan(x) + c$

b) $y = \dfrac{\cos(x)}{\sin(x)}$

$u = \cos(x)$ $\quad v = \sin(x)$

$du/dx = -\sin(x)$ $\quad dv/dx = \cos(x)$

$$dy/dx = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$= \frac{-1\,(\sin^2(x) + \cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\cos^2(x)$$

c) $(\tan(x) + \cot(x))^2 = \tan^2(x) + 2\tan(x)\cot(x) + \cot^2(x)$

$$= \tan^2(x) + 2\left(\frac{\sin(x)}{\cos(x)} \times \frac{\cos(x)}{\sin(x)}\right) + \cot^2(x)$$

$$= \tan^2(x) + 2 + \cot^2(x)$$

$$= \tan^2(x) + 1 + \cot^2(x) + 1$$

$$= \sec^2(x) + \operatorname{cosec}^2(x)$$

d) $$\int_{0.5}^{1} \sec^2(x) + \operatorname{cosec}^2(x)$$

$$= \left[\tan(x) - \frac{\cos(x)}{\sin(x)}\right]_{0.5}^{1} = \tan(1) - \frac{\cos(1)}{\sin(1)} - \tan(0.5) + \frac{\cos(0.5)}{\sin(0.5)}$$

From a)

From b)

$= 0.91531\ldots - -1.28418\ldots$

$= 2.1994\ldots$

$= 2.2$ (2sf)