Core 3 - June 2005

① a) $y = x \sin(2x)$

Product Rule:

$u = x \qquad v = \sin(2x)$

$du/dx = 1 \qquad dv/dx = 2\cos(2x)$

$dy/dx = \sin(2x) + 2x\cos(2x)$

b) i) $y = (x^2 - 6)^4$

Chain Rule:

$y = t^4 \qquad t = x^2 - 6$

$dy/dt = 4t^3 \qquad dt/dx = 2x$

$dy/dx = 2x \times 4t^3$

$= 8x(x^2-6)^3$

ii) $\int x(x^2-6)^3 \, dx$

$\int 8x(x^2-6)^3 \rightarrow (x^2-6)^4 + c$

So $\int x(x^2-6)^3 \rightarrow 1/8 \, (x^2-6)^4 + c$

② a) $h(x) = fg(x) = f\left(\frac{6}{x+3}\right) = \frac{6}{x+3} - 2$

b) i) $h^{-1}(x)$ ~~x~~ $y = \frac{6}{x+3} - 2$

$y + 2 = \frac{6}{x+3}$

$(y+2)(x+3) = 6$

$(x+3) = \frac{6}{y+2}$

$x = \frac{6}{y+2} - 3$

switch: $y = \frac{6}{x+2} - 3 = h^{-1}(x)$

ii) Range $h^{-1}(x)$ = domain $h(x)$

Range : $h^{-1}(x) \neq 3$

③ a) $\int e^{4x}\,dx \rightarrow \frac{1}{4}e^{4x} + c$

b) $\int e^{4x}(2x+1)\,dx$ $\qquad u = 2x+1 \qquad dv/dx = e^{4x}$

$du/dx = 2 \qquad v = \frac{1}{4}e^{4x}$

$uv - \int v\, du/dx$

$= \frac{1}{4}e^{4x}(2x+1) - \int \frac{1}{2}e^{4x}$

$= \frac{1}{4}e^{4x}(2x+1) - \frac{1}{8}e^{4x} + c$

c) $\int \frac{1+\ln(x)}{x}\,dx$ $\qquad u = 1 + \ln(x)$

$\int \frac{u}{x}\, x\,du$ $\qquad du/dx = 1/x$

$\rightarrow x\,du = dx$

$\rightarrow \int u\,du$

$= \frac{u^2}{2} + c \qquad = \frac{(1+\ln(x))^2}{2} + c$

④ a) $\tan^2(x) = \sec(x) + 11$ $\qquad \tan^2(x) = \sec^2(x) - 1$

$\sec^2(x) - 1 = \sec(x) + 11$

$\rightarrow \sec^2(x) - \sec(x) - 12 = 0$

b) $(\sec(x) - 4)(\sec(x) + 3) = 0$

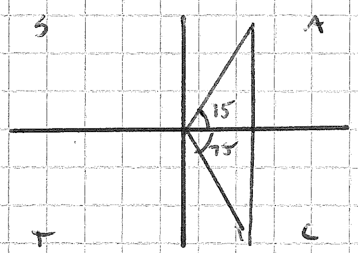
$\downarrow \qquad \downarrow$

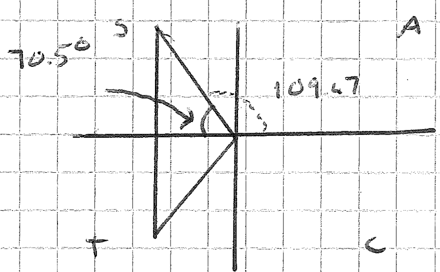
$\sec(x) = 4 \qquad \sec(x) = -3 \qquad \sec(x) = 1/\cos(x)$

$\rightarrow \cos(x) = 1/4 \qquad \cos(x) = -1/3$

c) $\cos(x) = 1/4 \qquad \cos(x) = -1/3$

$x = 75.52 \qquad x = 109.47..$





$x = 76°, 284° \qquad x = 109°, 251°$

(5) a) $2e^{x} = 5$

$e^{x} = 5/2$

$x = \ln(5/2)$

b) i) $2e^{x} + 5e^{-x} = 7$

$2e^{x} + \dfrac{5}{e^{x}} = 7$ $\qquad y = e^{x}$

$2y + 5/y = 7$

$2y^2 + 5 = 7y$

$\rightarrow 2y^2 - 7y + 5 = 0$

ii) $(2y - 5)(y - 1) = 0$

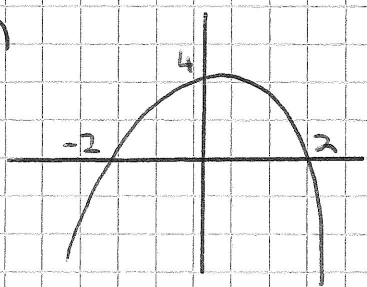
$2y - 5 = 0 \qquad\qquad y - 1 = 0$

$y = 5/2 \qquad\qquad y = 1$

$e^{x} = 5/2 \qquad\qquad e^{x} = 1$

$x = \ln(5/2) \qquad\qquad x = \ln(1) = 0$

(6) a) i)



ii) $V = \pi \int_0^2 y^2 \, dx$

$y = 4 - x^2$

$y^2 = (4 - x^2)^2$

$\pi \int (4 - x^2)^2 \, dx$

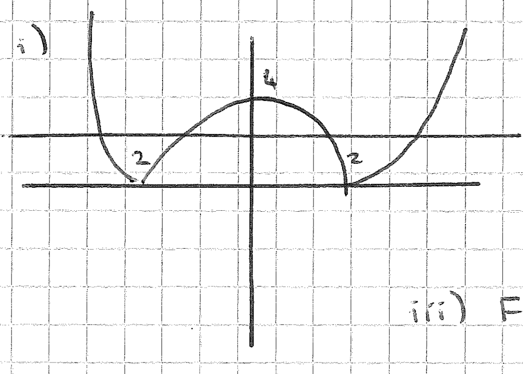
$= \pi \int 16 - 8x^2 + x^4 \, dx$

$= \pi \left[16x - 8x^3/3 + x^5/5 \right]_0^2$

$= \pi \left[16(2) - 8(2)^3/3 + (2)^5/5 - 0 \right]$

$= \dfrac{256\pi}{15}$

b) i)



ii) $|4 - x^2| = 3$

$4 - x^2 = 3 \quad$ or $\quad x^2 - 4 = 3$

$1 = x^2 \qquad\qquad x^2 = 7$

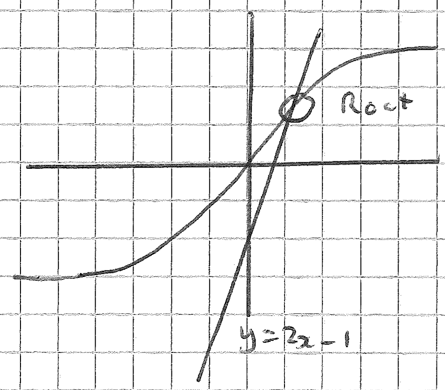
$x = 1$ or $-1 \qquad\qquad x = \sqrt{7}$ or $-\sqrt{7}$

iii) From graph: $|4 - x^2| < 3$

$-\sqrt{7} < x < -1, \qquad 1 < x < \sqrt{7}$

⑦ a) $y = \tan^{-1}(x)$

b) i) see graph



b) ii) $\tan^{-1}(x) = 2x - 1$

RADIANS!

$\rightarrow \tan^{-1}(x) - 2x + 1 = 0$

$x = 0.8 \rightarrow \tan^{-1}(0.8) - 2(0.8) + 1 \rightarrow 0.0747...$

$x = 0.9 \rightarrow \tan^{-1}(0.9) - 2(0.9) + 1 \rightarrow -0.067.$

change of sign, therefore root between 0.8 and 0.9

c) $x_1 = 0.8$

$x_2 = \frac{1}{2}(\tan^{-1}(0.8) + 1) = 0.83737...$

$x_3 = 0.84855. = 0.85 \; (2sf)$

⑧ a) $y = e^x \rightarrow y = e^{2x} + 3$

[2x] stretch parallel to x-axis, scale factor 1/2

[+3] translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

b)

x (midpoint)	y
2.25	93.017
2.75	247.692
3.25	668.142
3.75	1811.042
	2819.89

width $= 2/4 = 0.5$

Area $= 0.5 \times 2819.89$

$= 1409.92...$

$= 1410 \; (3sf)$

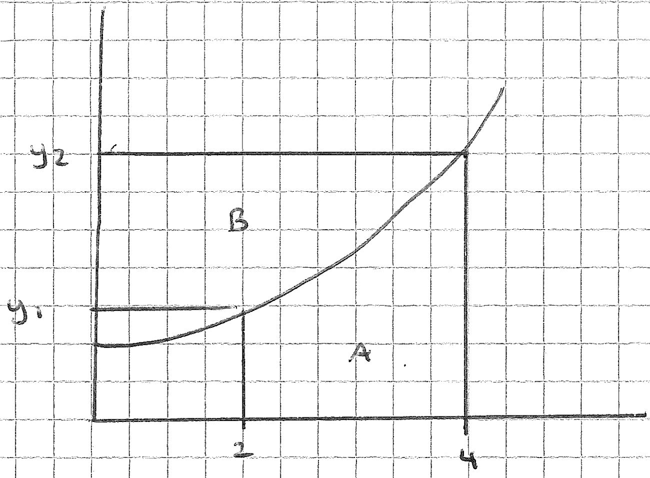
c) $\int_2^4 e^{2x} + 3 \, dx$

$= \left[\frac{1}{2} e^{2x} + 3x \right]_2^4$

$= \frac{1}{2} e^8 + 12 - \frac{1}{2} e^4 - 6$

$= \frac{1}{2}(e^8 - e^4) + 6$

d)



$y_1 = e^4 + 3$

$y_2 = e^8 + 3$

$$\begin{aligned}
\text{Area } (A + B) &= \text{Big Rectangle} - \text{Small Rectangle} \\
&= 4(e^8 + 3) - 2(e^4 + 3) \\
&= 4e^8 + 12 - 2e^4 - 6
\end{aligned}$$

$$\begin{aligned}
\text{Area of B} &= \text{Area } (A+B) - \text{Area } (A) \\
&= 4e^8 + 12 - 2e^4 - 6 - \tfrac{1}{2}(e^8 - e^4) - 6 \\
&= 4e^8 - \tfrac{1}{2}e^8 - 2e^4 + \tfrac{1}{2}e^4 \\
&= \tfrac{7}{2}e^8 - \tfrac{3}{2}e^4
\end{aligned}$$