# S3 • Using probability computer games

## Mathematical goals

To enable learners to:

- confront and overcome common misconceptions about probability;
- count equally likely outcomes, using diagrams;
- discuss relationships between theoretical probabilities, observed outcomes and sample sizes;
- calculate probabilities of dependent and independent events.

These goals may be adapted for learners aiming at lower levels. For example, you may decide to focus only on the first three goals.

## Starting points

Learners will not need any prior knowledge about probability in order to use and analyse the games. This knowledge will be developed through the activity and the discussion that follows it.

It would however be helpful to use session **S2 Evaluating probability statements** before this one. This session will allow you to revisit some of the misconceptions that are described in that session.

## Materials required

For each small group of learners you will need:

- Access to a computer loaded with the programs *Coin races* and *Dice races*;
- Sheet 1 – *Coin races*;
- Sheet 2 – *Recording sheet for coin races*;
- Sheet 3 – *Dice races*;
- Sheet 4 – *Recording sheet for dice races*.

Learners aiming at lower levels may find it more helpful to use just the first one or two games in each set.

## Time needed

This is flexible, depending on the number of games used. As a rough guide, allow 1 hour for *Coin races* and 1 hour for *Dice races*. It is not essential that learners try to complete all the games. It is better to tackle two or three in depth than to cover them all superficially.

## Suggested approach

### Beginning the session

If you have a data projector or interactive whiteboard, it is very helpful to work through one race with the whole group. Before doing so, however, ask learners to make predictions about who will win and ask them to explain their reasoning.

As you work through a race, show learners how to fill in the recording sheet for that race. In particular, show them how to record the positions of the horses as the winner crosses the finishing line. Rather than simply writing the finishing order (1, 2, 3 etc.) it is more interesting to write the positions (the number of crosses in each row shows this) as the winner finishes the course.

### Working in groups

Ask learners to work in pairs at each computer, making sure they have enough room to write down their results and their reasoning.

> Learners aiming at lower levels may find it helpful to work through some games using real dice and coins before they try to use the simulations.

Ask learners to work on one of the two situations, using the computer programs provided: *Coin races* or *Dice races*. The procedure for each set of situations is similar (see Sheets 1–4).

Learners are asked to:

1. Predict the outcome of each race before starting it. You may need to re-emphasise this as learners often find it hard to stop and think when they are working at a computer.
2. Run the race on the computer and record the outcome in a table. They should record the position of each horse when the winner reaches the finish. This should be repeated at least three times for each situation.
3. Reflect on what happened and try to explain this. For example:
   - Does the outcome vary very much from race to race?
   - Why does a particular horse seem to win more often than the others?
   - Why does a particular horse come last more often than the others?
   - Could the winning distance have been predicted?

## Whole group discussion

After a set of races, collect together the results from the whole group and hold a discussion about the results. In the notes that follow, we have outlined one direction in which the discussion might go.

*(i) Coin races*

For the two-coin race, learners will have found that horse 1 wins most often, though the results vary for each race. If the group results are aggregated and average positions are found as the winner finishes, learners will find that, when horse 1 wins, the others have reached approximately the half-way stage. The race seems unfair. Ask learners why this should be so.

At this point, someone may mention the equally likely outcomes HH, HT, TH and TT and that this shows why horse 1 is twice as likely to move as the other horses. Explain that these outcomes may be shown using a list, a two-way sample space table or a tree diagram:

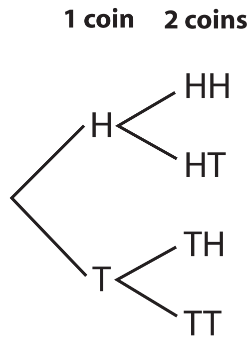
HH

HT

TH

TT

| | | Second coin | |
|---|---|---|---|
| | | **H** | **T** |
| First coin | **H** | HH | HT |
| | **T** | TH | TT |




Learners aiming at lower levels may find discussion of these representations helpful even if they do not go on to calculate probabilities.

Explain how this relates to the probabilities of each horse moving. Note that this is not the same as the probability of each horse winning. The longer the course, the more chance horse 1 has of winning. This relates to the 'sample size is irrelevant' misconception that is discussed in session **S2 Evaluating probability statements**.

Ask learners to draw tree diagrams for the three-horse situation in a similar way. They should then count the number of ways of getting 0, 1, 2 . . . heads for each number of coins and thus explain the likelihood of each horse winning.

They may like to list their observations and observe the patterns that emerge:

| | | Number of ways of getting | | | | | | Total outcomes |
|---|---|---|---|---|---|---|---|---|
| | | 0 Heads | 1 Heads | 2 Heads | 3 Heads | 4 Heads | 5 Heads | |
| Number of coins | 1 | 1 | 1 | | | | | 2 |
| | 2 | 1 | 2 | 1 | | | | 4 |
| | 3 | 1 | 3 | 3 | 1 | | | 8 |
| | 4 | 1 | 4 | 6 | 4 | 1 | | 16 |
| | 5 | 1 | 5 | 10 | 10 | 5 | 1 | 32 |

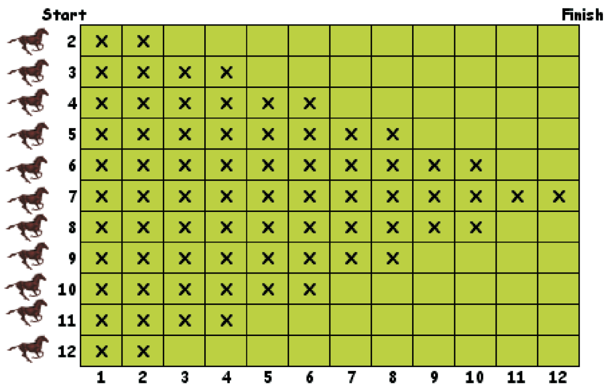
The probability of each horse winning can be calculated directly from this table. Thus with five coins, the probability that horse number 2 ('2 heads') wins is $\frac{10}{32} = \frac{5}{16}$. Learners may be able to predict further rows of this table from those shown here.

Relate this to the outcomes observed. Was it true that, for three coins, horses 2 and 3 won most often and were approximately three times as far down the course as the other two horses? What if the results of the whole group were aggregated?

*(ii) Dice races*

In the Sums race, learners should notice that the horses in the centre of the field tend to win more often than those at the outside. Can they explain this? Refer again to the sample space diagrams that display equally likely outcomes:

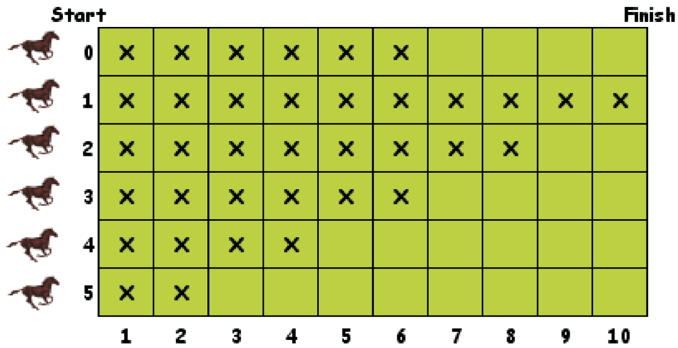
| Sums race | | First die | | | | | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 |
| Second die | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| | 6 | 7 | 8 | 9 | 10 | 11 | 12 |




The diagram on the left shows, for example, that we would expect horse 7 to move six times more often than horse 2. Show learners how to calculate, from this diagram, the probabilities that particular horses will move. Can learners describe how this table gives a prediction of the positions of horses when horse 7 finishes? (This is shown in the diagram on the right.)

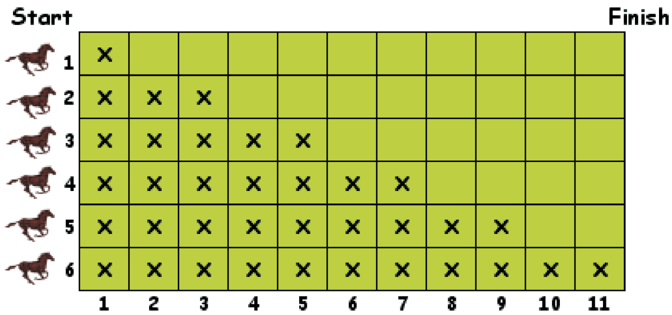
Now learners can be asked to analyse the remaining two situations in a similar manner, using sample space diagrams. They should find that the results from the Differences race give an asymmetric distribution, with horse 1 the most likely to win.

| Differences race | | First die | | | | | |
|---|---|---|---|---|---|---|---|
| | | ***1*** | ***2*** | ***3*** | ***4*** | ***5*** | ***6*** |
| **Second die** | ***1*** | 0 | 1 | 2 | 3 | 4 | 5 |
| | ***2*** | 1 | 0 | 1 | 2 | 3 | 4 |
| | ***3*** | 2 | 1 | 0 | 1 | 2 | 3 |
| | ***4*** | 3 | 2 | 1 | 0 | 1 | 2 |
| | ***5*** | 4 | 3 | 2 | 1 | 0 | 1 |
| | ***6*** | 5 | 4 | 3 | 2 | 1 | 0 |




They should find that the results from the Max dice race also give an asymmetric distribution, with horse 6 the most likely to win.

| Max race | | First die | | | | | |
|---|---|---|---|---|---|---|---|
| | | ***1*** | ***2*** | ***3*** | ***4*** | ***5*** | ***6*** |
| **Second die** | ***1*** | 1 | 2 | 3 | 4 | 5 | 6 |
| | ***2*** | 2 | 2 | 3 | 4 | 5 | 6 |
| | ***3*** | 3 | 3 | 3 | 4 | 5 | 6 |
| | ***4*** | 4 | 4 | 4 | 4 | 5 | 6 |
| | ***5*** | 5 | 5 | 5 | 5 | 5 | 6 |
| | ***6*** | 6 | 6 | 6 | 6 | 6 | 6 |




The Multiples dice race is slightly more complicated as a particular throw of the dice may result in two or more horses moving at one time. Thus a throw of $3 \times 4 = 12$ will result in horses 2, 3, 4 and 6 all moving. Ask learners to tell you which of horses 6 and 2 will be the faster and why.

The easiest place to begin is by simply writing out a list of possible products. Learners can then count the multiples systematically.

| Multiples race | | First die | | | | | |
|---|---|---|---|---|---|---|---|
| | | ***1*** | ***2*** | ***3*** | ***4*** | ***5*** | ***6*** |
| **Second die** | ***1*** | 1 | 2 | 3 | 4 | 5 | 6 |
| | ***2*** | 2 | 4 | 6 | 8 | 10 | 12 |
| | ***3*** | 3 | 6 | 9 | 12 | 15 | 18 |
| | ***4*** | 4 | 8 | 12 | 16 | 20 | 24 |
| | ***5*** | 5 | 10 | 15 | 20 | 25 | 30 |
| | ***6*** | 6 | 12 | 18 | 24 | 30 | 36 |

There are:

27 multiples of 2 and so prob (mult of 2) = $\frac{27}{36}$

20 multiples of 3 and so prob (mult of 3) = $\frac{20}{36}$

15 multiples of 4 and so prob (mult of 4) = $\frac{15}{36}$

11 multiples of 5 and so prob (mult of 5) = $\frac{11}{36}$

15 multiples of 6 and so prob (mult of 6) = $\frac{15}{36}$

Note that the probabilities do not add up to 1.

This enables us to predict the outcome: horse 2 should win, followed by horse 3, then 4 and 6 (together), with 5 bringing up the rear.

In each of the above situations, learners may question why their own results do not correspond to the theory. This is a good time to consider the issue of sample size. Small samples may not correspond to these results but, aggregated over the whole group, the results should correspond more closely.

### Reviewing and extending learning

Ask learners to analyse a game that they haven't yet used. For example, you could ask them to analyse a game where they have one coin and one die. If they throw a head, they double the number thrown on the die, otherwise the number stands. Can learners work out the possible outcomes and their probabilities?

## What learners might do next

Learners may enjoy developing their own GCSE probability question. A suitable session, **S7 Developing an exam question: probability**, is provided in this pack.

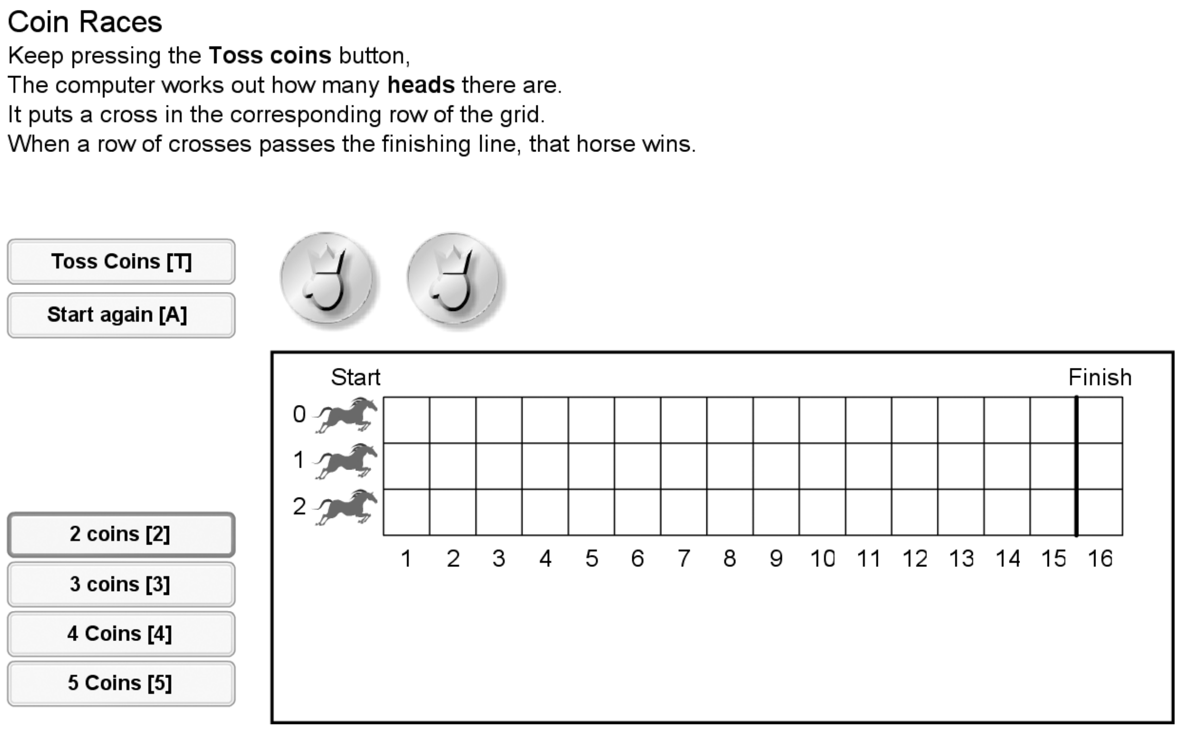
## Further ideas

This activity uses multiple representations to deepen understanding of probability. Learners may find it helpful to devise sample space and tree diagrams for their own dice, coin and spinner situations.

## S3 Sheet 1 – *Coin races*

You will be using the computer for these tasks.
Start with the two-coin race and work through the other races in order.
You should write your results in the tables on your recording sheet.

1. Before you start each race, write down the positions you predict horses will be in as the winner crosses the finishing line.
   Why do you think this?
   Write down your reason next to the table in the space provided.




2. Run each race several times.
   Each time, record the positions of the horses as the winner crosses the finishing line.
   (Count how many crosses there are in each row).

3. Describe and explain any patterns you find in your data.

Repeat steps 1–3 for the three-, four- and five-coin races.

## S3 Sheet 2 – *Recording sheet for coin races*

### 2 coin tosses

| | Horse number (number of heads thrown) | | |
|---|---|---|---|
| | 0 | 1 | 2 |
| *Your prediction* | | | |
| **First race** actual finishing positions | | | |
| **Second race** actual finishing positions | | | |
| **Third race** actual finishing positions | | | |

### 3 coin tosses

| | Horse number | | | |
|---|---|---|---|---|
| | 0 | 1 | 2 | 3 |
| *Your prediction* | | | | |
| **First race** actual finishing positions | | | | |
| **Second race** actual finishing positions | | | | |
| **Third race** actual finishing positions | | | | |

### 4 coin tosses

| | Horse number | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 |
| *Your prediction* | | | | | |
| **First race** actual finishing positions | | | | | |
| **Second race** actual finishing positions | | | | | |
| **Third race** actual finishing positions | | | | | |

### 5 coin tosses

| | Horse number | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| *Your prediction* | | | | | | |
| **First race** actual finishing positions | | | | | | |
| **Second race** actual finishing positions | | | | | | |
| **Third race** actual finishing positions | | | | | | |

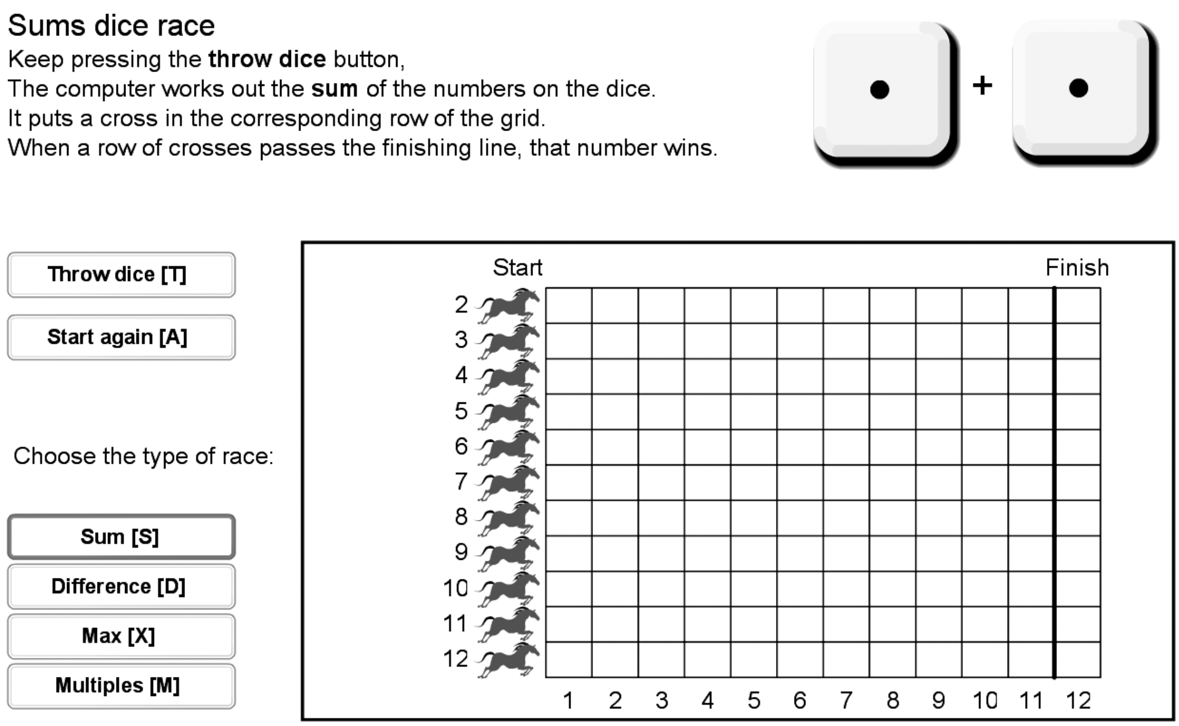
*Space for your reasoning*

In order to draw sensible conclusions, you will need to put these results together with those from the rest of the group.

## S3 Sheet 3 – *Dice races*

You will be using the computer for these tasks.
Start with the Sums race and then work through the other races in order.
You should write your results in the tables on your recording sheet.

1. Before you start each race, write down the positions you predict the horses will be in as the winner crosses the finishing line.
   Use the row of the table that is labelled 'Your prediction'.
   Why do you think this?
   Write down your reason next to the table in the space provided.




2. Run each race several times.
   Each time, record the positions of the horses as the winner crosses the line. (Count how many crosses there are in each row.)
3. Try to explain any patterns you find in your data.

Repeat steps 1, 2 and 3 for the Difference, Max and Multiples races.

## S3 Sheet 4 – *Recording sheet for dice races*

### Addition race

| | Horse number | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| *Your prediction* | | | | | | | | | | | |
| **First race** actual finishing positions | | | | | | | | | | | |
| **Second race** actual finishing positions | | | | | | | | | | | |
| **Third race** actual finishing positions | | | | | | | | | | | |

### Difference race

| | Horse number | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| *Your prediction* | | | | | | |
| **First race** actual finishing positions | | | | | | |
| **Second race** actual finishing positions | | | | | | |
| **Third race** actual finishing positions | | | | | | |

### Max race

| | Horse number | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| *Your prediction* | | | | | | |
| **First race** actual finishing positions | | | | | | |
| **Second race** actual finishing positions | | | | | | |
| **Third race** actual finishing positions | | | | | | |

### Multiples race

| | Horse number | | | | |
|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 |
| *Your prediction* | | | | | |
| **First race** actual finishing positions | | | | | |
| **Second race** actual finishing positions | | | | | |
| **Third race** actual finishing positions | | | | | |

*Space for your reasoning*

In order to draw sensible conclusions you will need to put these results together with those from the rest of the group.

If some learners feel more ambitious, they may like to try to develop spinners that are unfair, or use more than two spinners.

### Working in groups (2)

The new questions should be passed around the group to be answered by other learners. Where learners have difficulty answering questions, the question-writers should explain what they intended and act as a teacher, helping other learners to answer the questions.

Alternatively, some of the new questions can be photocopied for further sessions or for homework.

### Reviewing and extending learning

Finally, hold a whole group discussion about what has been learned, drawing out any continuing misconceptions. You should include a discussion of the level of difficulty of the new questions.

## What learners might do next

Ask learners to choose another question from an exam paper and follow the process adopted in this session.

(i) Answer the question;

(ii) Ask new questions about the same situation (and answer them);

(iii) Change the situation and write a new question.

## Further ideas

This method for developing exam questions can be used in any topic. Examples in this pack include:

**N10 Developing an exam question: number;**

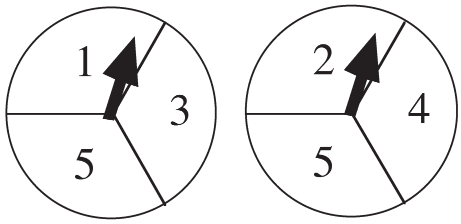
**A8 Developing an exam question: generalising patterns;**

**SS8 Developing an exam question: transformations.**

## S7 Sheet 1 – *Spinners*

Two fair spinners are numbered 1, 3, 5 and 2, 4, 5 respectively.

Spinner A Spinner B




You spin the spinners and add the numbers together.

- If the total is even, Amy wins a prize.
- If the total is a multiple of 3, Max wins a prize.
- If the total is 5 or 7, Sam wins a prize.

1. Draw a table to show all the equally likely outcomes:

| | | **Spinner A** | | |
|---|---|---|---|---|
| | | **1** | **3** | **5** |
| **Spinner B** | **2** | | | |
| | **4** | | | |
| | **5** | | | |

2. Complete the table below:

| **Name** | **Probability of winning a prize** |
|---|---|
| Amy | $\frac{1}{3}$ |
| Max | |
| Sam | |

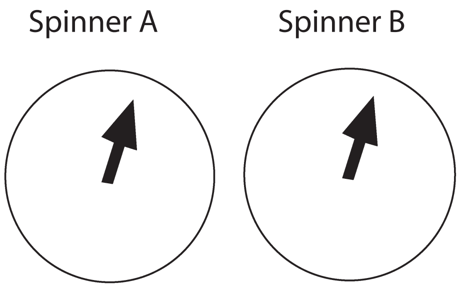
3. Explain what is wrong with the following statement:

   "The probability that someone will win is 1. This means that the probabilities in question 2 should add up to 1."

4. What further questions can you ask?

## S7 Sheet 2 – *Template for spinners question*

Two fair spinners are numbered ........................ and ..................... respectively.




You spin the spinners and ............................... the numbers together.

If the result is ............................ then .........................................

If the result is ............................ then .........................................

If the result is ............................ then .........................................

1. Draw a ..................................... to show all the equally likely outcomes:

2. ..........................................................................................................................................

3. ..........................................................................................................................................

# S8 • Using binomial probabilities

## Mathematical goals

To enable learners to:

- calculate binomial probabilities;
- calculate cumulative binomial probabilities.

To develop learners' understanding of:

- the context in which it is appropriate to use binomial probabilities;
- the symmetrical nature of the formula for a binomial probability, e.g. P(2 right out of 13) = P(11 wrong out of 13);
- alternative strategies for calculating cumulative binomial probabilities,
  e.g. P(at least 3 successes out of 10 trials) = 1 – P(fewer than 3 successes).

## Starting points

Learners should understand what is meant by independent events and how to calculate probabilities of a series of independent events using $P(A \text{ and } B) = P(A) \times P(B)$.

## Materials required

For each learner you will need:

- mini-whiteboard.

For each small group of learners you will need:

- Card set A – *Events*;
- Card set B – *Probabilities*.

## Time needed

At least 1 hour.

## Suggested approach

### Beginning the session

Discuss what independent events are and how to calculate the probability of successive independent events.

### Whole group discussion (1)

Use successively larger tree diagrams to establish the formula for binomial probabilities. Introduce the notation $\binom{12}{9}$ or ${}^{12}C_9$ and check that learners can use their calculators to evaluate it. Emphasise the criteria that must be true in order for the binomial to be applied. Ask learners to come up with some repeated events that are clearly not suitable for use with binomial probabilities e.g. the probability that it rains at midday and on successive days.

Give each learner a binomial probability such as ${}^{15}C_7\ 0.8^7 \times 0.2^8$ and ask them to write a question for which that is the answer. Collect in the questions and read them out one at a time. Ask learners to write the answer to each question on their whiteboards and compare these with the original answer. Discuss any differences and, in particular, any questions in which the information is not clear enough in order to arrive at the answer. Check that the events given in each question are appropriate for use with binomial probabilities.

### Working in groups

Arrange learners in pairs and give each pair Card set A – *Events*. Set the scenario:

> "The probability of winning a game is always 0.6 and there are 8 games left to play."

Ask learners to match any events on the cards that are essentially the same event, e.g. P(lose at least 5) and P(win fewer than 4). Ask pairs of learners to compare with other pairs how many matchings they have until there is a consensus in the group as a whole.

Encourage learners who find the language difficult to draw a line and highlight the events that the card refers to e.g.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Losses | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| Wins | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

This example represents P(win 3 or more) or P(lose 5 or less).

If learners find this easy, ask them to add some more phrases for the events in each group, e.g. P(win 2 or less) can be placed with P(win fewer than 3).

Give out Card set B – *Probabilities*. Ask learners to match the probabilities with their sets of events from Card set A.

## Whole group discussion (2)

Where there is more than one *Probabilities* card that matches a set of events, discuss the reason for this. Emphasise that it does not matter whether you consider the number of wins or the number of losses; the formulae give the same value.

Ask learners to find two events that, together, cover all possibilities, e.g. P(win more than 3) and P(win 3 or less). Discuss the possibility of calculating P(win more than 3) as 1 – P(win 3 or less) as a more efficient strategy.

Ask learners to identify other events with long probabilities that can be reduced in this way. Encourage learners to find the cards that contain these long probabilities and, for each one, write the alternative calculation on the card.

## Reviewing and extending learning

Building on Whole group discussion (1), give each pair of learners a cumulative binomial probability and ask them to write a suitable question for which it is the answer. As before, collect in the questions and read them out one at a time. Learners write down the answer to the question on their mini-whiteboards. These are then compared with the original answer. Discuss any differences and in particular any questions in which the information is not clear enough in order to give the answer. Check that the events given in each question are appropriate for use with cumulative binomial probabilities.

## What learners might do next

Use cumulative binomial tables for a larger number of repeated events.

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**S8 Card set A – *Events***

| | |
|---|---|
| P(win fewer than 3) | P(win no more than 3) |
| P(win at least 3) | P(lose more than 5) |
| P(win exactly 3) | P(win more than 3) |
| P(lose no more than 5) | P(lose at least 5) |
| P(lose exactly 5) | P(win fewer than 4) |
| P(lose fewer than 5) | P(lose more than 4) |
| P(lose 5 or less) | P(win 3 or more) |
| P(lose 5 or more) | P(win 3 or less) |

## S8 Card set B – *Probabilities*

| |
|---|
| $^8C_5 0.4^5 \times 0.6^3 + {}^8C_4 0.4^4 \times 0.6^4 + {}^8C_3 0.4^3 \times 0.6^5 + {}^8C_2 0.4^2 \times 0.6^6 + {}^8C_1 0.4 \times 0.6^7 + 0.6^8$ |
| $^8C_2 0.6^2 \times 0.4^6 + {}^8C_1 0.6 \times 0.4^7 + 0.4^8$ |
| $^8C_6 0.4^6 \times 0.6^2 + {}^8C_7 0.4^7 \times 0.6 + 0.4^8$ |
| $^8C_3 0.6^3 \times 0.4^5 + {}^8C_4 0.6^4 \times 0.4^4 + {}^8C_5 0.6^5 \times 0.4^3 + {}^8C_6 0.6^6 \times 0.4^2 + {}^8C_7 0.6^7 \times 0.4 + 0.6^8$ |
| $^8C_4 0.4^4 \times 0.6^4 + {}^8C_3 0.4^3 \times 0.6^5 + {}^8C_2 0.4^2 \times 0.6^6 + {}^8C_1 0.4 \times 0.6^7 + 0.6^8$ |
| $^8C_3 0.6^3 \times 0.4^5 + {}^8C_2 0.6^2 \times 0.4^6 + {}^8C_1 0.6 \times 0.4^7 + 0.4^8$ |
| $^8C_5 0.4^5 \times 0.6^3 + {}^8C_6 0.4^6 \times 0.6^2 + {}^8C_7 0.4^7 \times 0.6 + 0.4^8$ |
| $^8C_4 0.6^4 \times 0.4^4 + {}^8C_5 0.6^5 \times 0.4^3 + {}^8C_6 0.6^6 \times 0.4^2 + {}^8C_7 0.6^7 \times 0.4 + 0.6^8$ |
| $^8C_3 0.6^3 \times 0.4^5$ |
| $^8C_5 0.4^5 \times 0.6^3$ |