Mechanics 1

Revision Notes

July 2012

MECHANICS 1

1. Mathematical Models in Mechanics

Assumptions and approximations often used to simplify the mathematics involved:

a) rigid body is a particle,

b) no air resistance,

c) no wind,

d) force due to gravity remains constant,

e) *light* pulleys and *light* strings etc. have no mass,

f) rods are uniform - constant mass per unit length,

g) a lamina is a flat object of negligible thickness and of constant mass per unit area,

h) the earth's surface, although spherical, is usually modelled by a plane

i) surface is smooth - no friction.

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2. Vectors in Mechanics.

A **vector** is a quantity which has both *magnitude* and *direction*.

A **scalar** is just a number - it has no direction - e.g. mass, time, etc.

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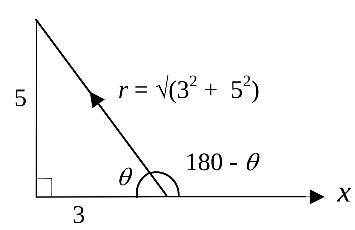
Magnitude and direction ⟵⟶ components

From component form, draw a sketch and use Pythagoras and trigonometry to find hypotenuse and angle.

Example:



$\mathbf{r} = -3\mathbf{i} + 5\mathbf{j} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ m

$\tan\theta = {}^{5}/_{3} \quad \Rightarrow \theta = 59.0^{o}$

$\Rightarrow$ $\mathbf{r}$ is a vector of magnitude $\sqrt{34}$ m, making an angle of 121.0^{o} with the x-axis.

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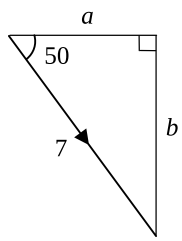
From magnitude and direction form, draw a sketch and use trigonometry to find x and y components.

Example:

$\boldsymbol{r}$ is a vector of length 7 *cm* making an angle of -50° with the x-axis.



$a = 7\cos 50^{\circ} = 4.50\ cm$,

$b = 7\ \sin 50^{\circ} = 5.36\ cm$

$\Rightarrow\ \boldsymbol{r} = \begin{bmatrix} 4.50 \\ -5.36 \end{bmatrix}\ cm.$

Example: Find a vector of length 20 in the direction of $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

Solution: Any vector in the direction of $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ must be a multiple, k, of $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

To find the multiple, k, we first find the length of $\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \sqrt{(-3)^2 + 4^2} = 5$

and as $20 = \mathbf{4} \times 5$ the multiple k must be **4**

so the vector of length 20 is $4 \times \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -12 \\ 16 \end{bmatrix}$

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Parallel vectors

Two vectors are parallel $\Leftrightarrow$ one is a multiple of another:

e.g. $6\mathbf{i} - 8\mathbf{j} = 2(3\mathbf{i} - 4\mathbf{j}) \quad \Leftrightarrow \quad 6\mathbf{i} - 8\mathbf{j}$ and $3\mathbf{i} - 4\mathbf{j}$ are parallel

Or $\begin{bmatrix} 6 \\ -8 \end{bmatrix} = 2 \times \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 6 \\ -8 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ are parallel.

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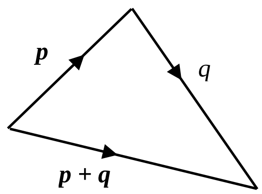
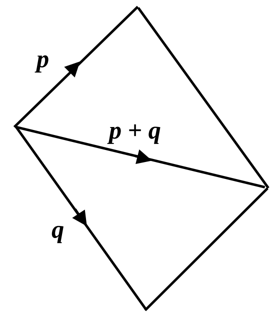
Adding vectors

Use a vector triangle or a vector parallelogram:



or



1) In component form: $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$

2) To add two vectors which are given in magnitude and direction form:

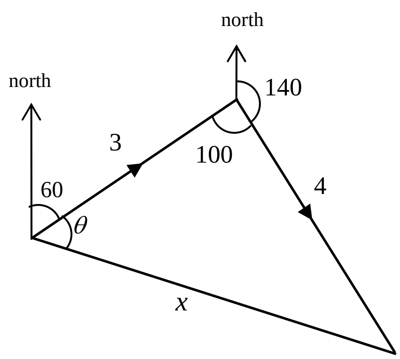
Either *a*) convert to component form, add and convert back,

Or *b*) sketch a vector triangle and use sine or cosine rule.

Example:

Add together, 3 miles on a bearing of 60^{o} and 4 miles on a bearing of 140^{o}.

Solution:



Using the cosine rule

$x^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 100 = 29.16756$

$\Rightarrow\; x = 5.4006996 = 5.40$

then, using the sine rule,

$\frac{4}{\sin\theta} = \frac{5.4006996}{\sin 100} \;\Rightarrow\; \sin\theta = 0.729393$

$\Rightarrow\; \theta = 46.83551 \quad \Rightarrow\;$ bearing $= 46.8 + 60 = 106.8^{o}$

Answer Resultant vector is 5.40 miles on a bearing of 106.8^{o}.

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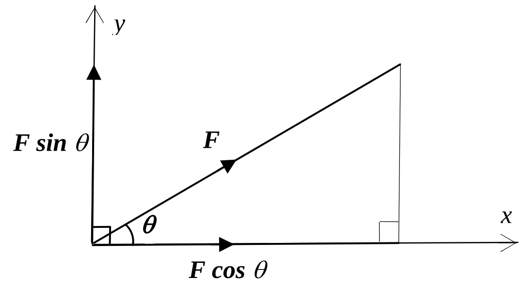
Resolving vectors in two perpendicular components

F has components $F \cos \theta$ and $F \sin \theta$ as shown.



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Vector algebra

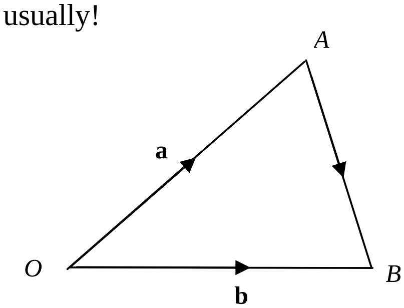
Notation, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, etc., but $\overrightarrow{OP} = \mathbf{r}$, usually!

To get **from** *A* **to** *B*

first go *A* to *O* using **–a**

then go *O* to *B* using **b**

$\Rightarrow \quad \overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$.



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Vectors in mechanics

Forces behave as vectors (the physicists tell us so) - *modelling*.

Velocity is a vector so must be given *either* in component form *or* as magnitude **and direction**.

Speed is the magnitude of the velocity so is a **scalar**.

Acceleration is a vector so must be given *either* in component form *or* as magnitude **and direction**.

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Velocity and displacement.

If a particle moves from the point (2, 4) with a constant velocity $\boldsymbol{v} = 3\mathbf{i} - 4\mathbf{j}$ for 5 seconds then its displacement vector will be *velocity* × *time* = $(3\mathbf{i} - 4\mathbf{j}) \times 5 = 15\mathbf{i} - 20\mathbf{j}$ and so its new position will be given by $(2\mathbf{i} + 4\mathbf{j}) + (15\mathbf{i} - 20\mathbf{j}) = 17\mathbf{i} - 16\mathbf{j}$.

Example: A particle is initially at the point (4, 11) and moves with velocity $\begin{bmatrix} 3 \\ -7 \end{bmatrix}$ m s^{-1}. Find its position vector after t seconds.

Solution: The displacement during t seconds will be $t \times \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

and so the new position vector will be $\boldsymbol{r} = \begin{bmatrix} 4 \\ 11 \end{bmatrix} + t \times \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} 4+3t \\ 11-7t \end{bmatrix}$.

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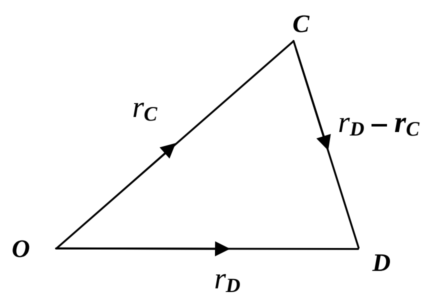
Relative displacement vectors

If you are standing at a point C and X is standing at a point D then the position vector of X relative to you is the vector $\overrightarrow{CD}$



and $\overrightarrow{CD} = \boldsymbol{r}_{D\ rel\ C} = \boldsymbol{d} - \boldsymbol{c} = \boldsymbol{r}_D - \boldsymbol{r}_C$

Thus if a particle A is at $\boldsymbol{r}_A = 3\mathbf{i} - 4\mathbf{j}$ and B is at $\boldsymbol{r}_B = 7\mathbf{i} + 2\mathbf{j}$

then the position of A relative to B is

$\overrightarrow{BA} = \boldsymbol{r}_{A\ rel\ B} = \boldsymbol{a} - \boldsymbol{b} = \boldsymbol{r}_A - \boldsymbol{r}_B = (3\mathbf{i} - 4\mathbf{j}) - (7\mathbf{i} + 2\mathbf{j}) = -4\mathbf{i} - 6\mathbf{j}$.

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Collision of moving particles

Example: Particle A is intially at the point (3, 4) and travels with velocity $9\mathbf{i} - 2\mathbf{j}$ m s^{-1}.
Particle B is intially at the point (6, 7) and travels with velocity $6\mathbf{i} - 5\mathbf{j}$ m s^{-1}.

(a) Find the position vectors of A and B at time t.

(b) Show that the particles collide and find the time and position of collision.

Solution:

(a) $\boldsymbol{r}_A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t \times \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 9t+3 \\ -2t+4 \end{bmatrix}$ (Initial position + displacement)

$\boldsymbol{r}_B = \begin{bmatrix} 6 \\ 7 \end{bmatrix} + t \times \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6t+6 \\ -5t+7 \end{bmatrix}$ (Initial position + displacement)

(b) If the particles collide then their x-coordinates will be equal

$\Rightarrow$ x-coords = $9t + 3 = 6t + 6$ $\Rightarrow$ $t = 1$

BUT we must also show that the y-coordinates are equal at $t = 1$.

y-coords = $-2t + 4 = -5t + 7$ $\Rightarrow$ $t = 1$.

$\Rightarrow$ particles collide when $t = 1$ at (12, 2).

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Closest distance between moving particles

Example: Two particles, A and B, are moving so that their position vectors at time t are

$$\mathbf{r}_A = \begin{bmatrix} 5-3t \\ t+2 \end{bmatrix} \text{ and } \mathbf{r}_B = \begin{bmatrix} 4-t \\ 2t+3 \end{bmatrix}.$$

(a) Find the position vector of B relative to A.

(b) Find the distance between A and B at time t.

(c) Find the minimum distance between the particles and the time at which this occurs.

Solution:

(a) $$\mathbf{r}_{B\ rel\ A} = \mathbf{r}_B - \mathbf{r}_A = \begin{bmatrix} 4-t \\ 2t+3 \end{bmatrix} - \begin{bmatrix} 5-3t \\ t+2 \end{bmatrix} = \begin{bmatrix} 2t-1 \\ t+1 \end{bmatrix} \quad \text{Answer.}$$

(b) The distance, d, between the particles is the length of

$$\overrightarrow{AB} = \mathbf{r}_{B\ rel\ A} = \begin{bmatrix} 2t-1 \\ t+1 \end{bmatrix}$$

$\Rightarrow$ $d^2 = (2t-1)^2 + (t+1)^2 = 4t^2 - 4t + 1 + t^2 + 2t + 1$
$= 5t^2 - 2t + 2$

$\Rightarrow$ $d = \sqrt{5t^2 - 2t + 2}$

(c) The minimum value of d will occur when the minimum value of d^2 occurs so we differentiate d^2 with respect to t.

$d^2 = 5t^2 - 2t + 2$

$\Rightarrow$ $\dfrac{d(d^2)}{dt} = 10t - 2 = 0$ for max and min $\Rightarrow t = 0.2$

and the second derivative $\dfrac{d^2(d^2)}{dt^2} = 10$ which is positive for $t = 0.2$

$\Rightarrow$ d^2 is a minimum at $t = 0.2$

$\Rightarrow$ minimum value of $d = \sqrt{5 \times 0.2^2 - 2 \times 0.2 + 2} = \sqrt{1.8} = 1.34$ to 3 S.F.

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Relative velocity

This is similar to relative position in that if C and D are at positions $\boldsymbol{r_C}$ and $\boldsymbol{r_D}$

then the position of D relative to C is $\overrightarrow{CD} = \boldsymbol{r_{D\,rel\,C}} = \boldsymbol{r_D} - \boldsymbol{r_C}$

which leads on to: -

if C and D are moving with velocities $\boldsymbol{v_C}$ and $\boldsymbol{v_D}$ then the velocity of D relative to C is

$$\boldsymbol{v_{D\,rel\,C}} = \boldsymbol{v_D} - \boldsymbol{v_C}.$$

Example: Particles A and B have velocities $\boldsymbol{v_A} = (12t - 3)\mathbf{i} + 4\mathbf{j}$ and $\boldsymbol{v_B} = (3t^2 - 1)\mathbf{i} + 2t\mathbf{j}$.

Find the velocity of A relative to B and show that this velocity is parallel to the x- axis for a particular value of t which is to be determined.

Solution: $\boldsymbol{v_A} = \begin{bmatrix} 12t - 3 \\ 4 \end{bmatrix}$ and $\boldsymbol{v_B} = \begin{bmatrix} 3t^2 - 1 \\ 2t \end{bmatrix}$

$\Rightarrow \quad \boldsymbol{v_{A\,rel\,B}} = \boldsymbol{v_A} - \boldsymbol{v_B} = \begin{bmatrix} 12t - 3t^2 - 2 \\ 4 - 2t \end{bmatrix}$

The y-coordinate = 0 for motion parallel to the x-axis $\Rightarrow$ $4 - 2t = 0$ when $t = 2$

$\Rightarrow$ the velocity is parallel to the x-axis when $t = 2$.

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3. Kinematics of a particle moving in a straight line or a plane.

Constant acceleration formulae.

$v = u + at$; $s = ut + ½at^2$; $s = ½(u + v)t$; $v^2 = u^2 + 2as$.

N.B. Units must be consistent - e.g. change $km\ h^{-1}$ to $m\ s^{-1}$ before using the formulae.

Example: A particle moves through a point O with speed 13 ms^{-1} with acceleration -6 $m\ s^{-2}$. Find the time(s) at which the particle is 12 m from O.

Solution:

$u = 13, \quad a = -6, \quad s = 12, \quad t = ?$.

Use $s = ut + ½at^2 \quad \Rightarrow \quad 12 = 13t + ½ \times (-6) \times t^2$

$\Rightarrow \quad 3t^2 - 13t - 12 = 0 \qquad \Rightarrow \quad (3t - 4)(t - 3) = 0$

$\Rightarrow \quad t = 1\,^1/_3$ or 3.

Answer Particle is 12 m from O after $1\,^1/_3$ or 3 seconds.

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Vertical motion under gravity

1] The acceleration always acts downwards whatever direction the particle is moving.

2] We assume that there is no air resistance, that the object is not spinning or turning and that the object can be treated as a particle.

3] We assume that the gravitational acceleration remains constant and is 9.8 m s^{-2}.

4] Always state which direction (up or down) you are taking as positive.

Example: A ball is thrown vertically upwards from O with a speed of 14 m s^{-1}.

(a) Find the greatest height reached.

(b) Find the total time before the ball returns to O.

(c) Find the velocity after 2 seconds.

Solution: Take upwards as the positive direction.

(a) At the greatest height, h, the velocity will be 0 and so we have

$u = 14$, $v = 0$, $a = -9.8$ and $s = h$ (the greatest height).

Using $v^2 - u^2 = 2as$ we have $0^2 - 14^2 = 2 \times (-9.8) \times h$

$\Rightarrow$ $h = 196 \div 19.6 = 10$.

Answer: Greatest height is 10 m.

(b) When the particle returns to O the distance, s, from O is 0 so we have

$s = 0$, $a = -9.8$, $u = 14$ and $t = ?$.

Using $s = ut + \frac{1}{2} at^2$ we have $0 = 14t - \frac{1}{2} \times 9.8t^2$

$\Rightarrow$ $t(14 - 4.9t) = 0$

$\Rightarrow$ $t = 0$ (at start) or $t = 2\,^6/_7$ seconds.

Answer: The ball takes $2\,^6/_7$ seconds to return to O.

(c) After 2 seconds, $u = 14$, $a = -9.8$, $t = 2$ and $v = ?$.

Using $v = u + at$ we have $v = 14 - 9.8 \times 2$

$\Rightarrow$ $v = -5.6$.

Answer: After 2 seconds the ball is travelling at 5.6 m s^{-1} **downwards**.

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Speed-time graphs

1] The **area** under a speed-time graph represents the **distance** travelled.

2] The **gradient** of a speed-time graph is the **acceleration**.

Example: A particle is initially travelling at a speed of 2 $m\ s^{-1}$ and immediately accelerates at 3 $m\ s^{-2}$ for 10 seconds; it then travels at a constant speed before decelerating at a 2 $m\ s^{-2}$ until it stops.

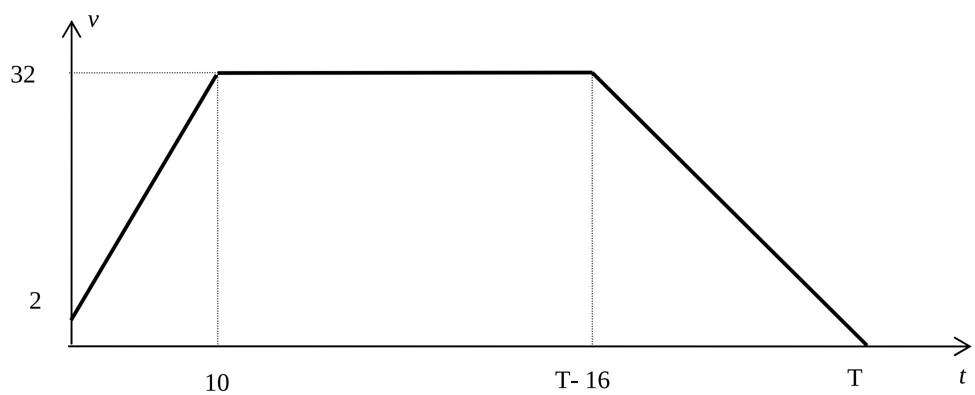
Find the maximum speed and the time spent decelerating: sketch a speed-time graph.

If the total distance travelled is 1130 metres, find the time spent travelling at a constant speed.

Solution:

For maximum speed: $u = 2$, $a = 3$, $t = 10$, $v = u + at \Rightarrow v = 32\ ms^{-1}$ is maximum speed.

For deceleration from 32 $m\ s^{-1}$ at 2 $m\ s^{-2}$ the time taken is $32 \div 2 = 16$ seconds.



Distance travelled in first 10 secs is area of trapezium $= \frac{1}{2}(2 + 32) \times 10 = 170$ metres,

distance travelled in last 16 secs is area of triangle $= \frac{1}{2} \times 16 \times 32 = 256$ metres,

$\Rightarrow$ distance travelled at constant speed $= 1130 - (170 + 256) = 704$ metres

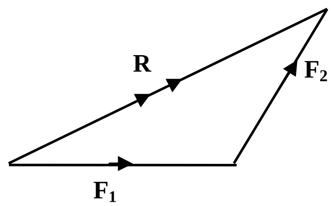
$\Rightarrow$ time taken at speed of 32 $m\ s^{-1}$ is $704 \div 32 = 22\ s$.

4. Statics of a particle.

Resultant forces

As forces are vectors you can add two forces geometrically using a triangle or a parallelogram.

R = F₁ + F₂



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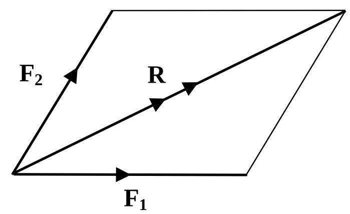
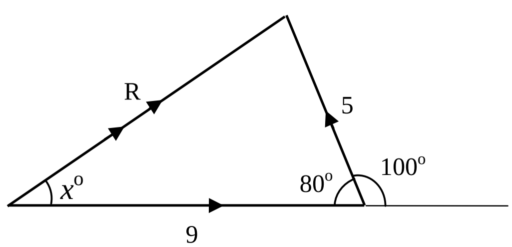


To find **R** from a diagram *either* draw accurately *or*, preferably, use sine and/or cosine rules.

Example: $\mathbf{F_1}$ and $\mathbf{F_2}$ are two forces of magnitudes 9 N and 5 N and the angle between their directions is 100^o. Find the resultant force.

Solution:



Using the cosine rule

$R^2 = 5^2 + 9^2 - 2 \times 5 \times 9 \times \cos 80$

$\Rightarrow$ R = 9.50640

and, using the sine rule,

$$\frac{5}{\sin x} = \frac{9.50640}{\sin 80} \quad \Rightarrow \quad x = 31.196^0$$

Answer: The resultant force is 9.51 *N* at an angle of 31.2^o with the 9 *N* force.

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Example:

Find the resultant of **P** = 5**i** - 7**j** and **Q** = -2**i** + 13**j**.

Solution:

R = **P** + **Q** = (5**i** - 7**j**) + (-2**i** + 13**j**) = 3**i** + 6j.

Answer Resultant is 3**i** + 6**j** N.

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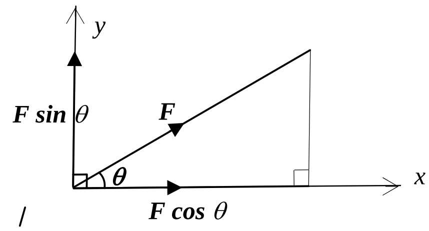
Resultant of three or more forces

Reminder:

We can resolve vectors in two perpendicular components as shown:



$\boldsymbol{F}$ has components $F \cos \theta$ and $F \sin \theta$.

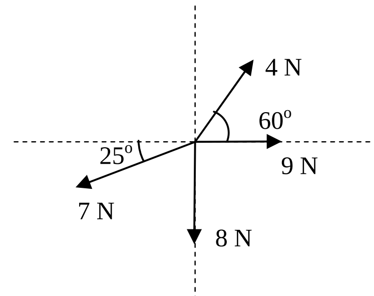
To find the resultant of three forces

1] convert into component form (**i** and **j**) , add and convert back

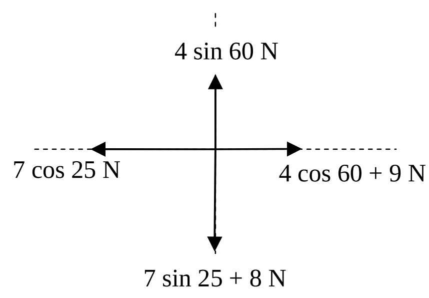
or 2] sketch a vector polygon and use sine/cosine rule to find the resultant of two, then combine this resultant with the third force to find final resultant.

For more than three forces continue with either of the above methods.

Example: Find the resultant of the four forces shown in the diagram.



Solution: First resolve the 7 N and 4 N forces horizontally and vertically



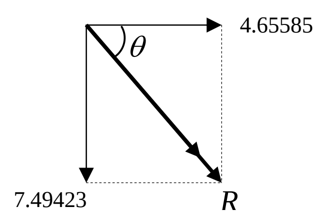
Resultant force $\longrightarrow$

is 4 cos 60 + 9 – 7 cos 25 = 4.65585 N

and resultant force $\downarrow$

is (7sin 25 + 8) – 4 sin 60 = 7.49423 N

giving this picture



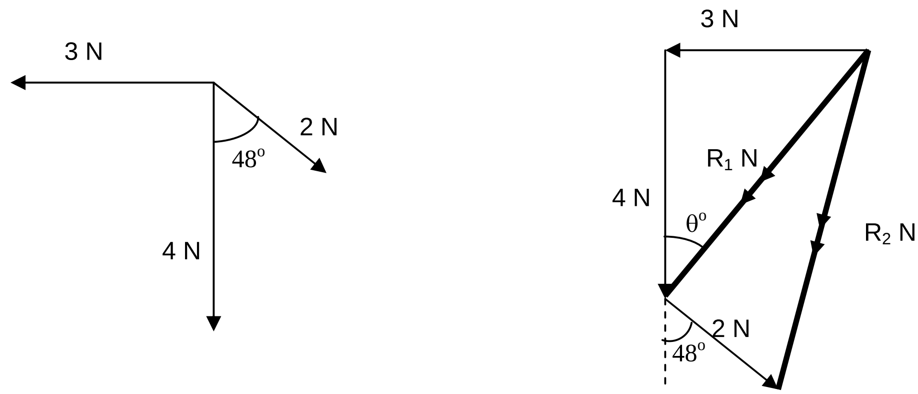
$\Rightarrow$ $R = \sqrt{4.65585^2 + 7.49423^2} = 8.82\ N$

and $\tan \theta = \dfrac{7.49423}{4.65585} \Rightarrow \theta = 58.1^{o}$

$\Rightarrow$ Answer resultant is 8.82 N at an angle of 58.1^0 below the 9 N force.

Example: Use a vector polygon to find the resultant of the three forces shown in the diagram.

Solution: To sketch the vector polygon, draw the forces end to end. I have started with the 3 N, then the 4 N and finally the 2 N force.



Combine the 3 N and 4 N forces to find the resultant **R_1** = 5 N with θ = 36.9°,

and now combine **R_1** with the 2 N force to find the final resultant **R_2** using the cosine and or sine rule.

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Equilibrium of a particle under coplanar forces.

If the sum of all the forces acting on a particle is zero (or if the resultant force is 0 N) then the particle is said to be in equilibrium.

Example: Three forces $\boldsymbol{P} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ N, $\boldsymbol{Q} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ N and $\boldsymbol{R} = \begin{bmatrix} a \\ b \end{bmatrix}$ N are acting on a particle which is in equilibrium.
Find the values of a and b.

Solution: As the particle is in equilibrium the sum of the forces will be **0** N.

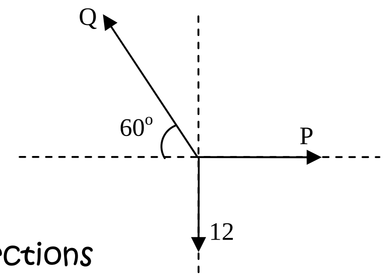
$\Rightarrow$ $\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{R} = \mathbf{0}$

$\Rightarrow$ $\begin{bmatrix} 7 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Answer: $a = -4$ and $b = -2$

Example: A particle is in equilibrium at O under the forces shown in the diagram.

Find the values of P and Q.

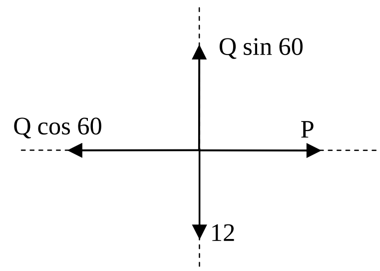


Solution:

First resolve Q in horizontal and vertical directions

Resolve ↑ $\Rightarrow$ Q sin 60 = 12 $\Rightarrow$ Q = 13.856.

Resolve ⟶ $\Rightarrow$ P = Q cos 60 = 6.928.



Answer P = 6.93 N and Q = 13.9 N

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Types of force

1) *Contact forces:* tension, thrust, friction, normal (i.e. perpendicular to the surface) reaction.

2) *Non-contact forces:* weight / gravity, magnetism, force of electric charges.

N.B. Never mark a force on a diagram without knowing what is providing it.

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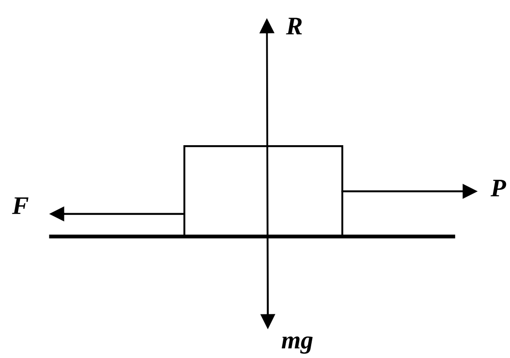
Friction

If we try to pull a box across the floor there is a **friction** force between the box and the floor.

If the box does not move the friction force will be **equal** to the force ***P***

and as ***P*** increases from 0 N the friction force will also increase from 0 N until it reaches its **maximum** value F_{max}, when the box will no longer be in equilibrium.

When friction force is at its maximum and the box is on the point of moving the box is said to be in **limiting equilibrium**.



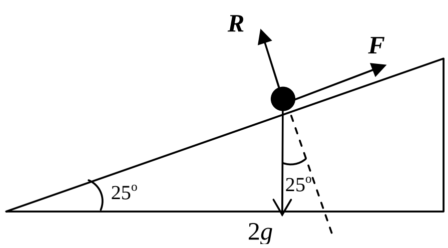
N.B. The **direction** of the **friction** force is **always opposite to** the **direction of motion** (or the direction in which the particle would move if there was no friction).

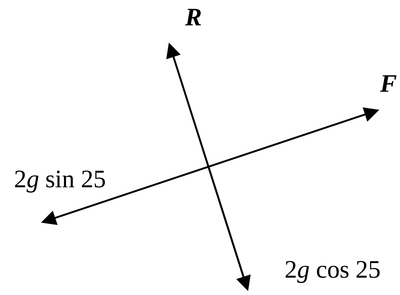
Example: A particle of mass 2 kg rests in equilibrium on a plane which makes an angle of 25° with the horizontal.

Find the magnitude of the friction force and the magnitude of the normal reaction.

Solution: First draw a diagram showing all the forces - the weight $2g$ N, the friction ***F*** N and the normal reaction ***R*** N. Remember that the particle would move down the slope without friction so friction must act **up** the slope.

Then **draw a second diagram** showing forces resolved along and perpendicular to the slope.





The particle is in equilibrium so

resolving perpendicular to the slope ***R*** $= 2g \cos 25 = 17.7636$,

and resolving parallel to the slope ***F*** $= 2g \sin 25 = 8.2833$.

Answer Friction force is 8.28 N and normal reaction is 17.8 N.

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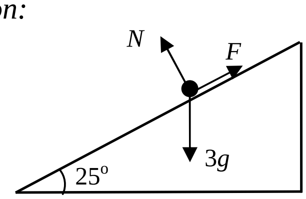
Coefficient of friction.

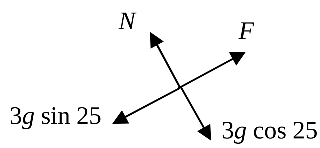
There is a maximum value, or *limiting* value, of the friction force between two surfaces. The ratio of this maximum friction force to the normal reaction between the surfaces is called the coefficient of friction.

$F_{max} = \mu N$, where μ is the coefficient of friction and N is the normal reaction.

Example: A particle of mass 3 *kg* lies in equilibrium on a slope of angle 25°. If the coefficient of friction is 0.6, show that the particle is in equilibrium and find the value of the friction force.

Solution:





Res ↖ $\Rightarrow$ $N = 3g \cos 25 = 26.645$

$\Rightarrow$ Maximum friction force is $F_{max} = \mu N = 0.6 \times 26.645 = 16.0$

Res ↗ $\Rightarrow$ $F = 3g \sin 25 = 12.4 < F_{max}$ if the particle is in equilibrium

Thus the friction needed to prevent sliding is 12.4 *N* and since the **maximum** possible value of the friction force is 16.0 *N* the particle will be in equilibrium and the actual friction force will be just 12.4 *N*.

Answer Friction force is 12.4 *N*.

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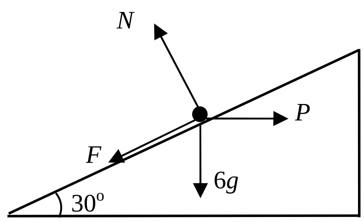
Limiting equilibrium

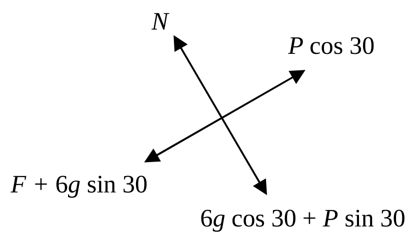
When a particle is in equilibrium but the friction force has reached its maximum or limiting value and is on the point of moving, the particle is said to be in *limiting equilibrium.*

Example: A particle of mass 6 *kg* on a slope of angle 30^o is being pushed by a horizontal force of *P N*. If the particle is in *limiting* equilibrium and is on the point of moving up the slope find the value of *P*, given that $\mu = 0.3$.

Solution:

As the particle is on the point of moving up the slope the friction force will be acting down the slope, and as the particle is in *limiting* equilibrium the friction force will be at its maximum or limiting value, $F = \mu N$.





Res ↖ $\Rightarrow$ $N = 6g \cos 30 + P \sin 30$, and $F = \mu N$

$\Rightarrow$ $F = 0.3N = 15.2767 + 0.15P$ —— **I**

Res ↗ $\Rightarrow$ $F + 6g \sin 30 = P \cos 30$ —— **II**

From **I** and **II**, $15.2767 + 0.15P + 6g \sin 30 = P \cos 30$

$\Rightarrow$ $P = 62.3819$

Answer $\underline{P = 62.4 \ N.}$

5. *Dynamics of a particle moving in a straight line.*

Newton's laws of motion.

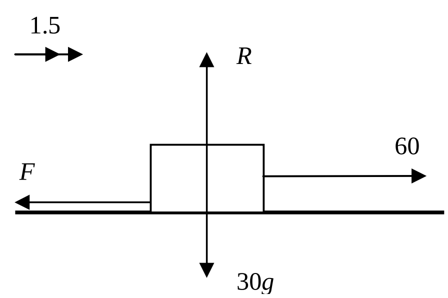
1) A particle will remain at rest or will continue to move with constant velocity in a straight line unless acted on by a resultant force.

2) For a particle with *constant* mass, *m kg*, the resultant force **F** *N* acting on the particle and its acceleration **a** $m\ s^{-2}$ satisfy the equation **F** = *m* **a**.

3) If a body *A* exerts a force on a body *B* then body *B* exerts an equal force on body *A* but in the opposite direction.

Example: A box of mass 30 kg is being pulled along the ground by a horizontal force of 60 N. If the acceleration of the trolley is 1.5 m s^{-2} find the magnitude of the friction force.

Solution: First draw a picture !!



No need to resolve as forces are already at 90^o to each other.

Resolve horizontally

$\Rightarrow$ $60 - F = 30 \times 1.5$

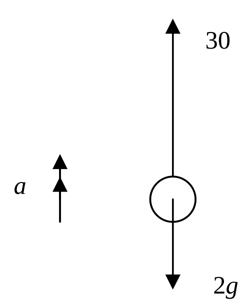
$\Rightarrow$ $F = 60 - 45 = 15.$

Answer <u>Friction force is 15 N</u>.

Example: A ball of mass 2 kg tied to the end of a string. The tension in the string is 30 N.

Find the acceleration of the ball and state in which direction it is acting.

Solution: First draw a picture!!



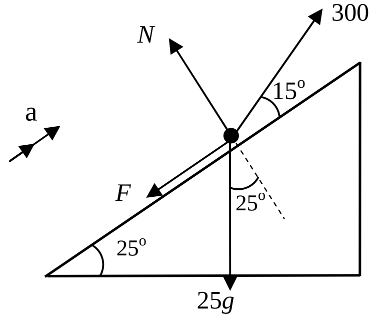
Resolve upwards $\Rightarrow$ $30 - 2g = 2a$

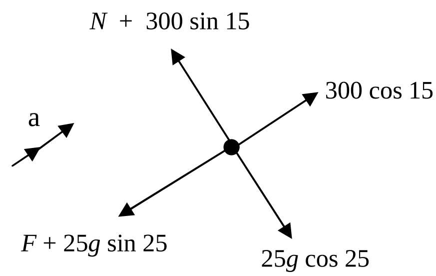
$\Rightarrow$ $a = 5.2$

Answer <u>Acceleration is 5.2 m s^{-2} upwards.</u>

Example: A particle of mass 25 *kg* is being pulled up a slope at angle of 25° above the horizontal by a rope which makes an angle of 15° with the slope. If the tension in the rope is 300 *N* and if the coefficient of friction between the particle and the slope is 0.25 find the acceleration of the particle.

Solution:





Res ↖ $\Rightarrow$ $N + 300 \sin 15 = 25g \cos 25$ $\Rightarrow$ $N = 144.39969$

and, since moving, friction is maximum $\Rightarrow$ $F = \mu N = 0.25 \times 144.39969 = 36.0999$

Res ↗ $\Rightarrow$ $300 \cos 15 - (F + 25g \sin 25) = 25a$

$\Rightarrow$ $a = 6.00545$

Answer the acceleration is 6.01 $m\ s^{-2}$.

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Connected particles

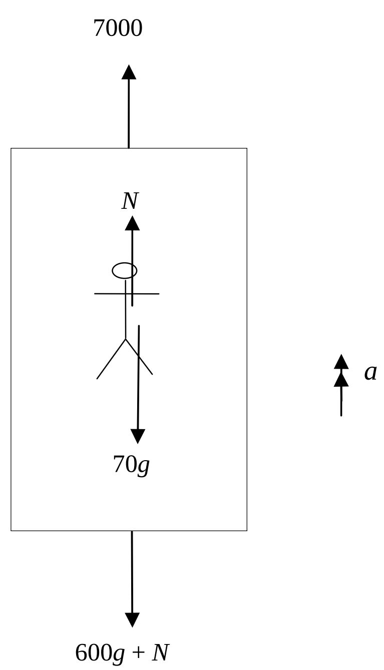
In problems with two or more connected particles, draw a **large** diagram in which the particles are clearly **separate**. Then put in all forces on each particle: don't forget Newton's third law there will be some 'equal and opposite' pairs of forces.

Example: A lift of mass 600 *kg* is accelerating upwards carrying a man of mass 70 *kg*. If the tension in the lift cables is 7000 N find the acceleration of the lift and the force between the floor and the man's feet.

Solution:

First draw a clear diagram with all forces on lift **and** all forces on man.

N.B. If the normal reaction on the man is N newtons then this means that the lift floor is pushing up on the man with a force of N newtons and therefore the man must be pushing down on the lift floor with an equal sized force of N newtons.



For the **lift**

Res $\uparrow$ $\quad 7000 - 600g - N = 600a$

For the **man**

Res $\uparrow$ $\quad N - 70g = 70a$

adding $\quad 7000 - 600g - 70g = 670a$

$\Rightarrow \quad a = 0.64776$

$\Rightarrow \quad N = 70g + 70a = 731.34$

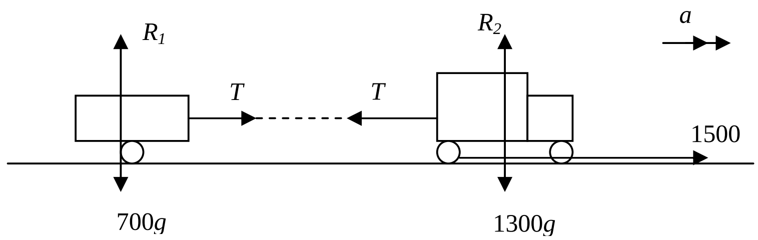
Answer: the acceleration is 0.648 $m\ s^{-2}$ and the force between the man and the floor is 731 N.

Example: A truck of mass 1300 kg is pulling a trailer of mass 700 kg. The driving force exerted by the truck is 1500 N and there is no resistance to motion.

Find the acceleration of the truck and trailer, and the force in the tow bar between the truck and the trailer.

Solution: First draw a picture!! separating the truck and the trailer to show the forces on each one.

If the force in the tow bar is T N then this force will be pulling the trailer and pulling back on the truck.



Note that the truck and trailer both have the same acceleration, assuming a rigid tow bar.

For the trailer

Resolve horizontally $\Rightarrow$ $T = 700a$

For the truck

Resolve horizontally $\Rightarrow$ $1500 - T = 1300a$

For truck and trailer as a single particle of mass 2000 kg

Resolve horizontally $\Rightarrow$ $1500 = 2000a$

$\Rightarrow$ $a = 0.75$

and from the trailer equation $T = 700a = 700 \times 0.75 = 525$

Answer <u>Acceleration is 0.75 m s^{-2} and force in tow bar is 525 N.</u>

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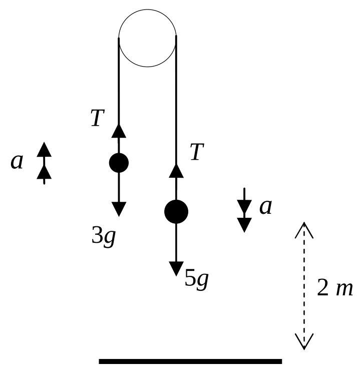
Particles connected by pulleys:

The string will always be *inextensible* and *light* and the pulley will always be *smooth* and *light*.

Example: Particles of mass 3 *kg* and 5 *kg* are attached to the ends of a light inextensible string which passes over a fixed smooth pulley. Both particles are initially 2 *m* above the floor.

The system is released from rest; find the greatest height of the lighter mass above the floor in the subsequent motion.

Solution:



Since the string is inextensible the accelerations of both particles will be equal in magnitude.

Since the string is light and the pulley is smooth the tensions on both sides will be equal in magnitude.

For 3 *kg* particle

Res $\uparrow$ $\Rightarrow$ $T - 3g = 3a$

For 5 *kg* particle

Res $\downarrow$ $\Rightarrow$ $5g - T = 5a$

adding $2g = 5a \Rightarrow a = 3.92$

Knowing that the acceleration of both particles is 3.92 $m\ s^{-2}$ we can now find the speed of the lighter particle when the heavier one hits the floor.

Both particles will have travelled 2 *m* and so, considering the 3 *kg* particle,

$\uparrow^{+}$ $u = 0$, $a = 3.92$, $s = 2$, $v = ?$ so using $v^2 = u^2 + 2as$

$v^2 = 2 \times 3.92 \times 2 = 15.68 \Rightarrow v = 3.9598.$

The remaining motion takes place freely under gravity as the string will have become slack when the heavier mass hit the deck!

$\uparrow^{+}$

$u = 3.9598$, $a = -g = -9.8$, $v = 0$, $s = ?$ so using $v^2 = u^2 + 2as$

$0 = 3.9598^2 + 2 \times -9.8 \times s \Rightarrow s = 0.8$

The lighter mass was originally 2 *m* above the floor, then moved up a further 2 *m* before the heavier mass hit the floor and then moved up a further 0.8 *m* after the string became slack.

Answer the lighter particle reached a height of 4.8 *m* above the floor.

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Impulse and Momentum.

a) We know that the velocity $\boldsymbol{v}$ $m\ s^{-1}$ of a body of mass m kg moving with a constant acceleration $\mathbf{a}$ $m\ s^{-2}$ for time t seconds is given by

$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$, where $\boldsymbol{u}$ is the initial velocity.

$\Rightarrow$ $m\boldsymbol{v} = m\boldsymbol{u} + m\boldsymbol{a}t$

$\Rightarrow$ $m\boldsymbol{a}t = m\boldsymbol{v} - m\boldsymbol{u}$.

Newton's Second Law states that $\boldsymbol{F} = m\boldsymbol{a}$

$\Rightarrow$ $\mathbf{F}t = m\boldsymbol{a}t$

$\Rightarrow$ $\boldsymbol{F}t = m\boldsymbol{v} - m\boldsymbol{u}$.

Note that $\boldsymbol{F}$ must be **constant** since **a** is constant.

b) We define the *impulse* of a constant force $\boldsymbol{F}$ N acting for a time t seconds to be $\boldsymbol{F}t$ Newton-seconds (Ns).

c) We define the *momentum* of a body of mass m kg moving with velocity $\boldsymbol{v}$ $m\ s^{-1}$ to be $m\boldsymbol{v}$ $kg\ m\ s^{-1}$.

d) The equation $\boldsymbol{F}t = m\boldsymbol{v} - m\boldsymbol{u}$ of paragraph (a) can now be thought of as

Impluse = Change in Momentum.

N.B. *Impulse* and *Momentum* are *vectors*.

Example: A ball of mass 2 kg travelling in a straight line at 4 m s^{-1} is acted on by a force of 3 N acting in the direction of motion for 5 secs.

Solution: The impulse of the force is $3 \times 5 = 15$ Ns in the direction of motion.

Taking the direction of motion as positive we have I = 15, $u = 4$, $m = 2$ and $v = ?$.

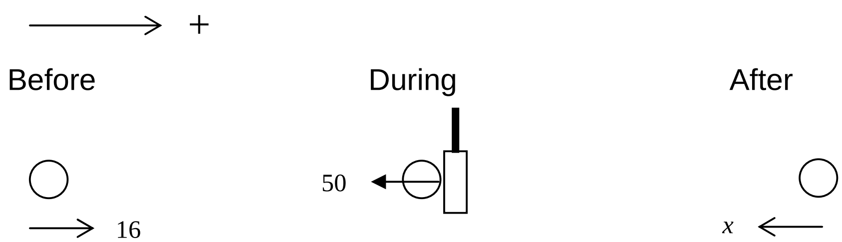
Using $I = mv - mu$ we have $15 = 2v - 2 \times 4$

$\Rightarrow$ $v = 11\frac{1}{2}$

Answer <u>speed after 5 seconds is 11 ½ m s^{-1}.</u>

Example: A ball of mass 1.5 kg is struck by a bat in the opposite direction to the motion of the ball. Before the impulse the ball is travelling at 16 m s^{-1} and the impulse of the bat on the ball is 50 Ns. Find the velocity of the ball immediately after impact.

Solution: Take the direction of motion of the ball as positive and let the speed after impact be x m s^{-1}.



$\Rightarrow$ $I = -50$, $u = 16$ and $v = -x$

Using $I = mv - mu$

$\Rightarrow$ $-50 = 1.5 \times (-x) - 1.5 \times 16$

$\Rightarrow$ $x = 17\ ^1/_3$.

Answer <u>velocity after impact is $17\ ^1/_3$ m s^{-1} away from bat.</u>

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Internal and External Forces and Impulses.

a) If a hockey ball is hit by a hockey stick then the impulse on the ball is an *external* impulse on the ball.

b) If two hockey balls collide then the impulses between the balls at the moment of collision are *internal* when considering the two balls *together*.
If we were considering *just one ball* then the impulse of collision would be *external* to that ball.

c) If an explosion separates a satellite from a rocket then the impulses of the explosion are *internal* when considering the rocket and the satellite *together*.
If we were considering the *satellite alone* then the impulse of the explosion would be *external* to the satellite.

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Conservation of momentum.

If there are no external impulses acting on a system then the total momentum of that system is conserved (i.e. remains the same at different times).

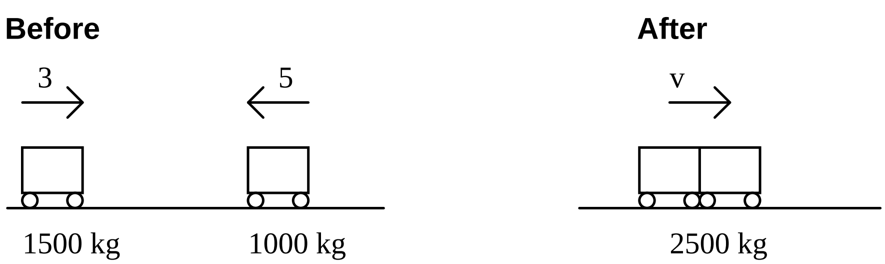
or **total momentum before impact equals total momentum after impact.**

Note that if there is an external impulse acting on the system then the momentum *perpendicular* to that impulse is conserved.

Example: A railway truck of mass 1500 *kg* is travelling in a straight line at 3 ms^{-1}. A second truck of mass 1000 *kg* is travelling in the opposite direction at 5 ms^{-1}. They collide (without breaking up) and couple together. With what speed and in what direction are they moving?

Solution: There is no external impulse (the impulse of gravity is ignored as the time interval is very short) and so momentum is conserved.
First draw diagrams!! **before** and **after**

Let the common speed after impact be v ms^{-1} in the direction of the velocity of the 1500 *kg* truck (if this direction is wrong then v will be negative):



Taking motion to the right as the positive direction, ⟶ +

Momentum **before** = $m_1u_1 + m_2u_2 = 1500 \times 3 + 1000 \times (-5) = -500$

Momentum **after** = $m_1v_1 + m_2v_2 = 1500\,v + 1000\,v = 2500\,v$

But momentum is conserved ⇒ $-500 = 2500\,v$

⇒ $v = -{}^1/_5 = -0.2$ ms^{-1}.

Answer <u>Speed is 0.2 $m\,s^{-1}$ in the direction of the 1000 *kg* truck's initial velocity.</u>

Example: Two balls *A* and *B* are travelling towards each other with speeds $u_A = 5$ m s^{-1} and $u_B = 6$ m s^{-1}.

After impact *A* is now travelling in the opposite direction at 3 m s^{-1}, and *B* continues to travel in its original direction but with speed 2 m s^{-1}.

The mass of *A* is 2 kg. Find the mass of *B*.

Solution: Let the mass of *B* be *m* kg.

First draw diagrams!! **before** and **after.**

Taking left as positive + ←

No external impulse ⇒ momentum conserved

⇒ total momentum before = total momentum after

⇒ $2 \times (-5) + m \times 6 = 2 \times 3 + m \times 2$

⇒ $4m = 16$ ⇒ $m = 4$

Answer Mass of ball *B* is 4 kg.

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Impulse in string between two particles

If a string links two particles which are moving apart then the string will become taut and, at that time, there will be an impulse in the string.

In this case the impulses on the two particles will be equal in magnitude but opposite in direction.

Thus when considering the two particles as one system there is no **external** impulse and the problem can be treated in the same way as collisions.

The assumptions involved are that the string is *light* (mass can be ignored) and *inextensible* (does not stretch).

Example: Two particles P and Q of masses 2 kg and 5 kg are connected by a light inextensible string. They are moving away from each other with speeds $u_P = 3 \text{ m s}^{-1}$ and $u_Q = 4 \text{ m s}^{-1}$.

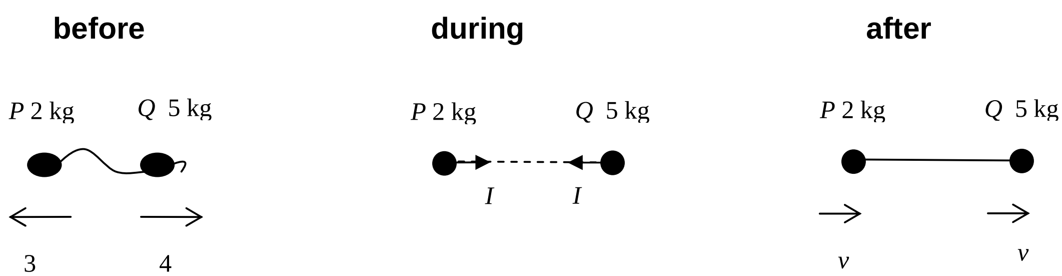
After the string becomes taut the particles move on with a constant velocity.

(a) Find this common velocity.

(b) Find the impulse in the string.

Solution: First draw a diagram!! showing **before, during** and **after**.

Let common speed be v



Taking direction to the right as positive $\longrightarrow$ +

(a) No external impulse $\Rightarrow$ total momentum conserved

$\Rightarrow$ $2 \times (-3) + 5 \times 4 = 2 \times v + 5 \times v$

$\Rightarrow$ $v = 2$

(b) To find impulse consider only **one** particle, P.

For particle P using $I = mv - mu$

$\Rightarrow$ $I = 2 \times v - 2 \times (-3)$ but $v = 2$

$\Rightarrow$ $I = 10$

Answer <u>Common speed is 2 m s^{-1} and Impulse = 10 Ns</u>

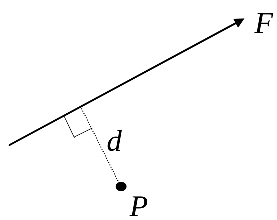
6. *Moments*

Moment of a Force

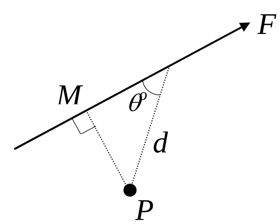
Definition: The **moment** of a force **F** about a point P is the product of the magnitude of **F** and the perpendicular distance from P to the line of action of the force.

Moments are measured in newton-metres, *Nm* and the *sense* - clockwise or anti-clockwise should always be given.

So:



or



moment = $F \times d$ clockwise

moment = $F \times PM = F \times d \sin\theta$ clockwise

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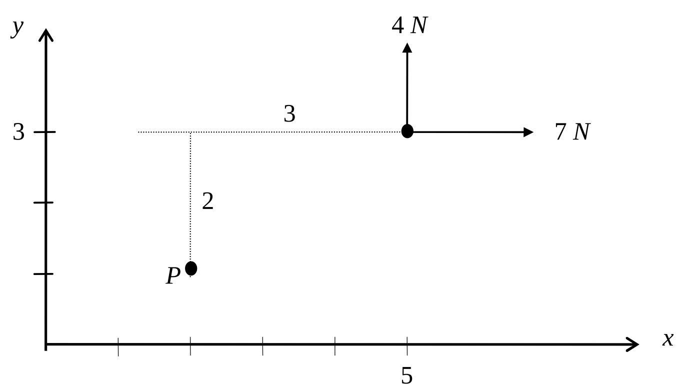
Sum of moments

Example: The force $7\mathbf{i} + 4\mathbf{j}$ N acts at the point (5, 3); find its moment about the point (2, 1)

Solution:

First draw a sketch showing the components of the force and the point (2, 1).



Taking moments about P *clockwise*

moment = $7 \times 2 - 4 \times 3 = 2$ Nm clockwise.

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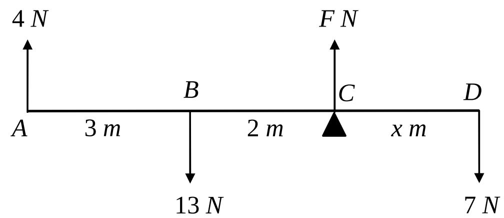
Moments and Equilibrium

If several forces are in equilibrium then

(i) The resultant force must be zero, and

(ii) The sum of the moments of all the forces about **any** point must be zero.

Example: If the forces in the diagram are in equilibrium find the force ***F*** and the distance x m.



Solution:

(i) The resultant force must be 0 $\Rightarrow$ $4 + F = 13 + 7$ $\Rightarrow$ $F = 16$

(ii) The sum of moments about **any** point must be 0.

Taking moments *clockwise* about C (as F acts through C its moment will be 0)

$\Rightarrow$ $4 \times 5 - 13 \times 2 + F \times 0 + 7 \times x = 0$ $\Rightarrow$ $x = 6 \div 7 = 0.857$

Answer $F = 16\ N$ and $x = 0.857\ m$.

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Non-uniform rods

A uniform rod has its centre of mass at its mid-point. A non-uniform rod (e.g. a tree trunk) would not necessarily have its centre of mass at its mid-point

Example: A non-uniform rod *AB* of mass 25 kg is of length 8 metres. Its centre of mass is 3 metres from *A*. The rod is pivoted about *M*, its mid point.

A mass of 20 kg is placed at *P* so that the system is in equilibrium. How far should this mass be from the end *A*?

What is now the reaction at the pivot?

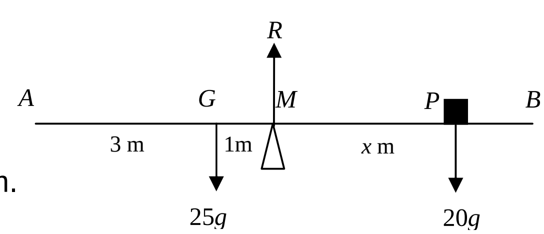
Solution: First draw a diagram!! showing **all** the forces.

Let 20 kg mass be x m from M.

Moments about the pivot

$\Rightarrow$ $1 \times 25g = x \times 20g$

$\Rightarrow$ $x = 1.25$ $\Rightarrow$ $AP = 5.25$ m.



Resolving vertically for equilibrium $\Rightarrow$ $R = 25g + 20g = 45g$

Answer 20 kg mass should be placed 5.25 m from A, and reaction is $45g$ N.

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Tilting rods

If a rod is supported at two points *A* and *B* then when the rod is about to tilt about *B* the normal reaction at *A* will be 0.

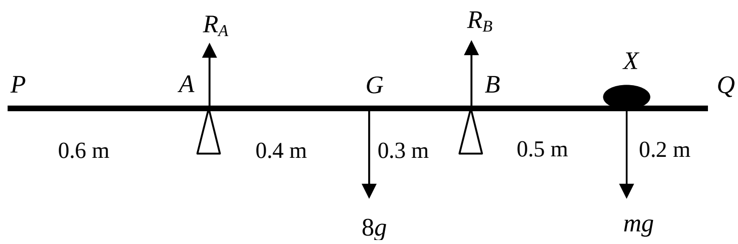
Example: A uniform plank *PQ* rests on two supports at *A* and *B*.

PQ = 2 m, *PA* = 0.6 m and *AB* =0.7 m. A mass *m* kg is placed at *X* on the rod between *B* and *Q* at a distance of 0.5 m from *B*.

The rod is on the point of tilting about *B*: find the value of *m*.

Solution: First draw a diagram!! showing **all** the forces.

The centre of mass, *G*, will be at mid point, *PG* = 1 m.



If the rod is on the point of tilting about *B* then the reaction at *A* will be 0

$\Rightarrow$ $R_A = 0$.

The system is in equilibrium so moments about any point will be 0. We could find the value of R_B but if we take moments about *B* the moment of R_B is 0, whatever the value of R_B.

Moments about *B*, taking clockwise as positive

$\Rightarrow$ $0.7 \times R_A - 0.3 \times 8g + 0 \times R_B + 0.5 \times mg$ (remember $R_A = 0$)

$\Rightarrow$ $m = 4.8$

Answer <u>Mass required to tilt rod is 4.8 kg.</u>

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