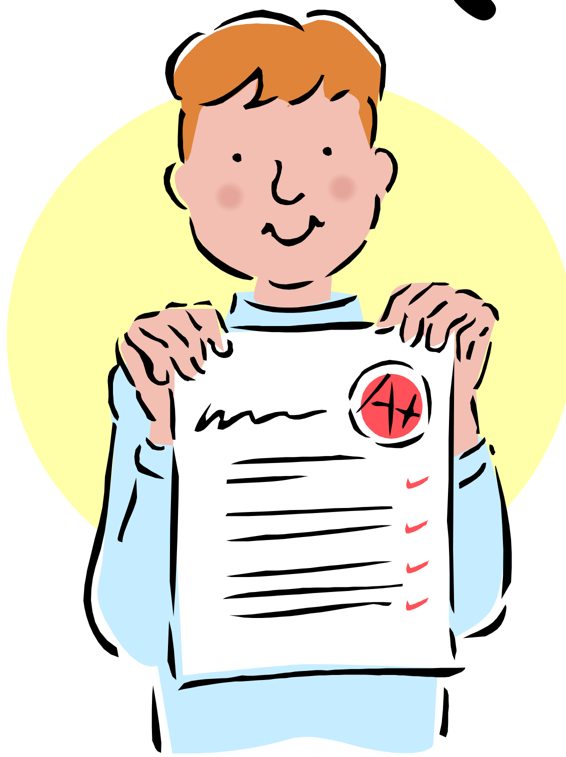
Core 3 (A2)



Practice Examination Questions

Mr A Slack

Trigonometric Identities and Equations

Trigonometry	I know what secant; cosecant and cotangent graphs look like and can identify appropriate restricted domains.	
	I know and can use the relationship between secant and cosine.	
	I know and can use the relationship between cosecant and sine.	
	I know and can use the relationship between cotangent and tangent.	
	I know how arcsin, arccos and arctan relate to sine, cosine and tangent.	
	I understand the graphs of arcsin, arcos and arctan.	
	I know and can use the identity: $\sin^2\theta + \cos^2\theta = 1$	
	I know and can use the identity: $\sec^2\theta = 1 + \tan^2\theta$.	
	I know and can use the identity: $\text{cosec}^2\theta = 1 + \cot^2\theta$.	
	I can work with angles in both degrees and radians.	
	I know and can use $\sin(A \pm B)$ and related $\sin(2A)$.	
	I know and can use $\cos(A \pm B)$ and related $\cos(2A)$.	
	I know and can use $\tan(A \pm B)$ and related $\tan(2A)$.	

1. (*a*) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \tan^2\theta \equiv \sec^2\theta$.

(2)

(*b*) Solve, for $0 \le \theta < 360°$, the equation

$$2\tan^2\theta + \sec\theta = 1,$$

giving your answers to 1 decimal place.

(6)

2. (*a*) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

(5)

(*b*) Given that $\sin\theta = \frac{\sqrt{3}}{4}$, find the exact value of $\sin 3\theta$.

(2)

3. (*a*) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that the $\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$.

(2)

(*b*) Hence, or otherwise, prove that

$$\operatorname{cosec}^4\theta - \cot^4\theta \equiv \operatorname{cosec}^2\theta + \cot^2\theta.$$

(2)

(*c*) Solve, for $90° < \theta < 180°$,

$$\operatorname{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta.$$

(6)

4. (*a*) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.

(2)

(*b*) Solve, for $0 \le \theta < 180°$, the equation

$$2\cot^2\theta - 9\operatorname{cosec}\theta = 3,$$

giving your answers to 1 decimal place.

(6)

5. (*a*) Show that

(i) $\dfrac{\cos 2x}{\cos x + \sin x} \equiv \cos x - \sin x, \qquad x \neq (n - \tfrac{1}{4})\pi,\ n \in \mathbb{Z},$

(2)

(ii) $\tfrac{1}{2}(\cos 2x - \sin 2x) \equiv \cos^2 x - \cos x \sin x - \tfrac{1}{2}$.

(3)

(*b*) Hence, or otherwise, show that the equation

$$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \frac{1}{2}$$

can be written as

$$\sin 2\theta = \cos 2\theta.$$

(3)

(*c*) Solve, for $0 \leq \theta < 2\pi$,

$$\sin 2\theta = \cos 2\theta,$$

giving your answers in terms of π.

(4)

6. (*a*) Given that $\cos A = \tfrac{3}{4}$, where $270° < A < 360°$, find the exact value of $\sin 2A$.

(5)

(*b*) (i) Show that $\cos\left(2x + \dfrac{\pi}{3}\right) + \cos\left(2x - \dfrac{\pi}{3}\right) \equiv \cos 2x$.

(3)

Given that

$$y = 3\sin^2 x + \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(2x - \frac{\pi}{3}\right),$$

(ii) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x$.

(4)

7. (i) Prove that

$$\sec^2 x - \operatorname{cosec}^2 x \equiv \tan^2 x - \cot^2 x.$$

(3)

(ii) Given that

$$y = \arccos x, \quad -1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi,$$

(*a*) express arcsin *x* in terms of *y*.

(2)

(*b*) Hence evaluate arccos *x* + arcsin *x*. Give your answer in terms of π.

(1)

8. (i) Without using a calculator, find the exact value of

$$(\sin 22.5° + \cos 22.5°)^2.$$

You must show each stage of your working.

(5)

(ii) (*a*) Show that $\cos 2\theta + \sin\theta = 1$ may be written in the form

$$k \sin^2\theta - \sin\theta = 0, \text{ stating the value of } k.$$

(2)

(*b*) Hence solve, for $0 \le \theta < 360°$, the equation

$$\cos 2\theta + \sin\theta = 1.$$

(4)

9. (*a*) Prove that

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \tan\theta, \qquad \theta \ne 90n°, \quad n \in \mathbb{Z}.$$

(4)

(*b*) Hence, or otherwise,

(i) show that $\tan 15° = 2 - \sqrt{3}$,

(3)

(ii) solve, for $0 < x < 360°$,

$$\operatorname{cosec} 4x - \cot 4x = 1.$$

(5)

Differentiation

Differentiation	I know and can use the differential of e^x.	
	I know and can use the differential of ln(x).	
	I know and can use the differential of sin(x).	
	I know and can use the differential of cos(x).	
	I know and can use the differential of tan(x).	
	I can use the chain rule to differentiate composite functions.	
	I can use the product rule to differentiate products.	
	I can use the quotient rule to differentiate fractions.	
	I can derive and use the differential of cosec(x).	
	I can derive and use the differential of sec(x).	
	I can derive and use the differential of cot(x).	
	I understand that $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ when working with eg. $\frac{dy}{dx}$ for $x = \sin 3y$.	

1. (*a*) Differentiate with respect to x

(i) $3\sin^2 x + \sec 2x$,

(3)

(ii) $\{x + \ln(2x)\}^3$.

(3)

Given that $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2}$, $x \neq 1$,

(*b*) show that $\dfrac{dy}{dx} = -\dfrac{8}{(x-1)^3}$.

(6)

2. Differentiate, with respect to x,

(a) $e^{3x} + \ln 2x$,

(3)

(*b*) $(5 + x^2)^{\frac{3}{2}}$.

(3)

3. (*a*) Differentiate with respect to x

(i) $x^2 e^{3x+2}$,

(4)

(ii) $\dfrac{\cos(2x^3)}{3x}$.

(4)

(*b*) Given that $x = 4\sin(2y + 6)$, find $\dfrac{dy}{dx}$ in terms of x.

(5)

4. The point P lies on the curve with equation $y = \ln\left(\frac{1}{3}x\right)$. The x-coordinate of P is 3.

Find an equation of the normal to the curve at the point P in the form $y = ax + b$, where a and b are constants.

(5)

4. The curve C has equation $x = 2 \sin y$.

(a) Show that the point $P\left(\sqrt{2}, \frac{\pi}{4}\right)$ lies on C.

(1)

(b) Show that $\frac{dy}{dx} = \frac{1}{\sqrt{2}}$ at P.

(4)

(c) Find an equation of the normal to C at P. Give your answer in the form $y = mx + c$, where m and c are exact constants.

(4)

5. (i) The curve C has equation $y = \frac{x}{9 + x^2}$.

Use calculus to find the coordinates of the turning points of C.

(6)

(ii) Given that $y = (1 + e^{2x})^{\frac{3}{2}}$, find the value of $\frac{dy}{dx}$ at $x = \frac{1}{2}\ln 3$.

(5)

6. A curve C has equation

$$y = e^{2x} \tan x, \quad x \neq (2n+1)\frac{\pi}{2}.$$

(*a*) Show that the turning points on C occur where $\tan x = -1$.

(6)

(*b*) Find an equation of the tangent to C at the point where $x = 0$.

(2)

7. (*a*) Differentiate with respect to x,

(i) $e^{3x}(\sin x + 2\cos x)$,

(3)

(ii) $x^3 \ln(5x+2)$.

(3)

Given that $y = \dfrac{3x^2+6x-7}{(x+1)^2}$, $x \neq -1$,

(*b*) show that $\dfrac{dy}{dx} = \dfrac{20}{(x+1)^3}$.

(5)

(*c*) Hence find $\dfrac{d^2y}{dx^2}$ and the real values of x for which $\dfrac{d^2y}{dx^2} = -\dfrac{15}{4}$.

(3)

8. The curve C has equation

$$y = (2x-3)^5$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(*a*) the value of w,

(2)

(*b*) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants.

(5)

9. (i) Differentiate with respect to x

(*a*) $y = x^3 \ln 2x$,

(*b*) $y = (x + \sin 2x)^3$.

(6)

Given that $x = \cot y$,

(ii) show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{1+x^2}$.

(5)

10. Differentiate with respect to x

(*a*) $\ln (x^2 + 3x + 5)$,

(2)

(*b*) $\frac{\cos x}{x^2}$.

(3)

Algebraic Fractions

Algebraic Fractions	I can simplify algebraic fractions by factorising and cancelling.	
	I can perform addition and subtraction with algebraic fractions.	

Mr A Slack

1. Express

$$\frac{2x^2+3x}{(2x+3)(x-2)} - \frac{6}{x^2-x-2}$$

as a single fraction in its simplest form.

(7)

2. (*a*) Simplify $\frac{3x^2-x-2}{x^2-1}$.

(3)

(*b*) Hence, or otherwise, express $\frac{3x^2-x-2}{x^2-1} - \frac{1}{x(x+1)}$ as a single fraction in its simplest form.

(3)

3. Given that

$$\frac{2x^4-3x^2+x+1}{(x^2-1)} \equiv (ax^2+bx+c) + \frac{dx+e}{(x^2-1)},$$

find the values of the constants a, b, c, d and e.

(4)

4.

$$f(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}, \quad x \neq -2.$$

(*a*) Show that $f(x) = \frac{x^2+x+1}{(x+2)^2}$, $x \neq -2$.

(4)

(*b*) Show that $x^2 + x + 1 > 0$ for all values of x.

(3)

(*c*) Show that $f(x) > 0$ for all values of x, $x \neq -2$.

(1)

5. $$h(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}, \quad x \geq 0.$$

(*a*) Show that $h(x) = \dfrac{2x}{x^2+5}$.

(4)

(*b*) Hence, or otherwise, find $h'(x)$ in its simplest form.

(3)

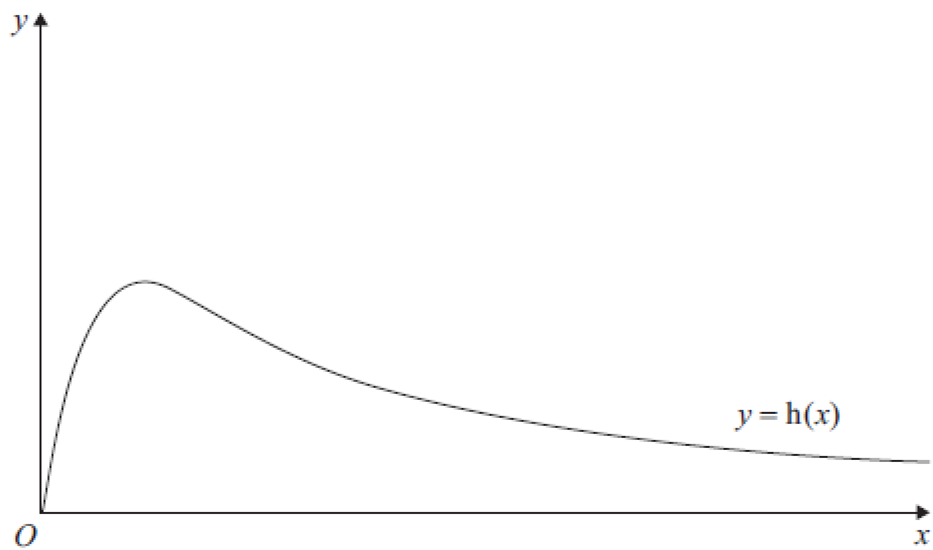


Figure 2

Figure 2 shows a graph of the curve with equation $y = h(x)$.

(*c*) Calculate the range of $h(x)$.

(5)

6. $$f(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}, \quad x \neq \pm 3, \; x \neq -\frac{1}{2}.$$

(*a*) Show that

$$f(x) = \frac{5}{(2x+1)(x+3)}.$$

(5)

The curve C has equation $y = f(x)$. The point $P\left(-1, -\frac{5}{2}\right)$ lies on C.

(*b*) Find an equation of the normal to C at P.

(8)

Functions

Algebra and Functions	I know the definition of a function and how it may be notated.	
	I know the meaning of 'one-to-one' and 'many-to-one' functions.	
	I understand and can use the "domain" of a function.	
	I understand and can use the "range" of a function.	
	I can work with composite functions.	
	I can find the inverse of a function.	
	I can draw the graph of an inverse function.	

1. The function f is defined by

$$\text{f}: x \mapsto \frac{5x+1}{x^2+x-2} - \frac{3}{x+2}, \quad x > 1.$$

(*a*) Show that $\text{f}(x) = \dfrac{2}{x-1}, \quad x > 1.$

(4)

(*b*) Find $\text{f}^{-1}(x)$.

(3)

The function g is defined by

$$\text{g}: x \mapsto x^2 + 5, \quad x \in \mathbb{R}.$$

(*b*) Solve $\text{fg}(x) = \frac{1}{4}$.

(3)

2. The functions f and g are defined by

$$\text{f}: x \mapsto 2x + \ln 2, \qquad x \in \mathbb{R},$$

$$\text{g}: x \mapsto \text{e}^{2x}, \qquad x \in \mathbb{R}.$$

(*a*) Prove that the composite function gf is

$$\text{gf}: x \mapsto 4\text{e}^{4x}, \qquad x \in \mathbb{R}.$$

(4)

(*b*) Sketch the curve with equation $y = \text{gf}(x)$, and show the coordinates of the point where the curve cuts the y-axis.

(1)

(*c*) Write down the range of gf.

(1)

(*d*) Find the value of x for which $\dfrac{\text{d}}{\text{d}x}[\text{gf}(x)] = 3$, giving your answer to 3 significant figures.

(4)

3. The function f is defined by

$$f : x \mapsto \ln(4 - 2x), \quad x < 2 \text{ and } x \in \mathbb{R}.$$

(*a*) Show that the inverse function of f is defined by

$$f^{-1} : x \mapsto 2 - \tfrac{1}{2}e^{x}$$

and write down the domain of f^{-1}.

(4)

(*b*) Write down the range of f^{-1}.

(1)

(*c*) Sketch the graph of $y = f^{-1}(x)$. State the coordinates of the points of intersection with the x and y axes.

(4)

The graph of $y = x + 2$ crosses the graph of $y = f^{-1}(x)$ at $x = k$.

The iterative formula

$$x_{n+1} = -\tfrac{1}{2}e^{x_n}, \quad x_0 = -0.3,$$

is used to find an approximate value for k.

(*d*) Calculate the values of x_1 and x_2, giving your answer to 4 decimal places.

(2)

(*e*) Find the values of k to 3 decimal places.

(2)

4. The functions f and g are defined by

$$f : x \mapsto 1 - 2x^3, \quad x \in \mathbb{R}.$$

$$g : x \mapsto \frac{3}{x} - 4, \quad x > 0, \; x \in \mathbb{R}.$$

(*a*) Find the inverse function f^{-1}.

(2)

(*b*) Show that the composite function gf is

$$gf : x \mapsto \frac{8x^3 - 1}{1 - 2x^3}.$$

(4)

(*c*) Solve gf $(x) = 0$.

(2)

(*d*) Use calculus to find the coordinates of the stationary point on the graph of $y = gf(x)$.

(5)

5. The function f is defined by

$$f : x \mapsto \frac{2(x-1)}{x^2 - 2x - 3} - \frac{1}{x-3}, \quad x > 3.$$

(*a*) Show that $f(x) = \dfrac{1}{x+1}, \quad x > 3.$

(4)

(*b*) Find the range of f.

(2)

(*c*) Find $f^{-1}(x)$. State the domain of this inverse function.

(3)

The function g is defined by

$$g : x \mapsto 2x^2 - 3, \quad x \in \mathbb{R}.$$

(*d*) Solve $fg(x) = \dfrac{1}{8}$.

(3)

6. The function f is defined by

$$f: x \mapsto 4 - \ln(x + 2), \quad x \in \mathbb{R}, \quad x \geq -1.$$

(*a*) Find $f^{-1}(x)$.
(3)

(*b*) Find the domain of f^{-1}.
(1)

The function g is defined by

$$g: x \mapsto e^{x^2} - 2, \quad x \in \mathbb{R}.$$

(*c*) Find fg(x), giving your answer in its simplest form.
(3)

(*d*) Find the range of fg.
(1)

Iteration

Numerical Methods	I can identify the location of roots of f(x)=0 by considering a change of sign of f(x).	
	I can find an approximate solution to an equation using simple iterative methods including relations of the form $x_{n+1} = f(x_n)$.	

1. $$f(x) = 3e^x - \tfrac{1}{2}\ln x - 2, \quad x > 0.$$

(*a*) Differentiate to find $f'(x)$.

(3)

The curve with equation $y = f(x)$ has a turning point at P. The x-coordinate of P is α.

(*b*) Show that $\alpha = \tfrac{1}{6}e^{-\alpha}$.

(2)

The iterative formula

$$x_{n+1} = \tfrac{1}{6}e^{-x_n}, \quad x_0 = 1,$$

is used to find an approximate value for α.

(*c*) Calculate the values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimal places.

(2)

(*d*) By considering the change of sign of $f'(x)$ in a suitable interval, prove that $\alpha = 0.1443$ correct to 4 decimal places.

(2)

2. $$f(x) = 2x^3 - x - 4.$$

(*a*) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{1}{2}\right)}.$$

(3)

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

(*b*) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{1}{2}\right)},$$

with $x_0 = 1.35$, to find, to 2 decimal places, the value of x_1, x_2 and x_3.

(3)

The only real root of $f(x) = 0$ is α.

(*c*) By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places.

(3)

3. **Figure 2**

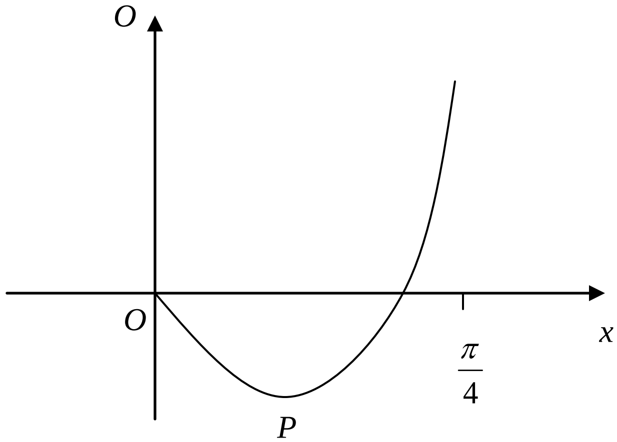


Figure 2 shows part of the curve with equation

$$y = (2x - 1)\tan 2x, \quad 0 \le x < \frac{\pi}{4}.$$

The curve has a minimum at the point *P*. The *x*-coordinate of *P* is *k*.

(*a*) Show that *k* satisfies the equation

$$4k + \sin 4k - 2 = 0.$$

(6)

The iterative formula

$$x_{n+1} = \tfrac{1}{4}(2 - \sin 4x_n), \quad x_0 = 0.3,$$

is used to find an approximate value for *k*.

(*b*) Calculate the values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimals places.

(3)

(*c*) Show that $k = 0.277$, correct to 3 significant figures.

(2)

4. $$f(x) = \ln(x + 2) - x + 1, \qquad x > -2, x \in \mathbb{R}.$$

(*a*) Show that there is a root of f(*x*) = 0 in the interval 2 < *x* < 3.

(2)

(*b*) Use the iterative formula

$$x_{n+1} = \ln(x_n + 2) + 1, \quad x_0 = 2.5,$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 5 decimal places.

(3)

(*c*) Show that *x* = 2.505 is a root of f(*x*) = 0 correct to 3 decimal places.

(2)

5. $$f(x) = 3x^3 - 2x - 6.$$

(*a*) Show that f (*x*) = 0 has a root, *α*, between *x* = 1.4 and *x* = 1.45.

(2)

(*b*) Show that the equation f (*x*) = 0 can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{2}{3}\right)}, \quad x \neq 0.$$

(3)

(*c*) Starting with $x_0 = 1.43$, use the iteration

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{2}{3}\right)}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places.

(3)

(*d*) By choosing a suitable interval, show that *α* = 1.435 is correct to 3 decimal places.

(3)

6. $$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

7. $$f(x) = 2\sin(x^2) + x - 2, \qquad 0 \leq x < 2\pi.$$

(*a*) Show that $f(x) = 0$ has a root α between $x = 0.75$ and $x = 0.85$.

(2)

The equation $f(x) = 0$ can be written as $x = [\arcsin(1 - 0.5x)]^{\frac{1}{2}}$.

(*b*) Use the iterative formula

$$x_{n+1} = [\arcsin(1 - 0.5x_n)]^{\frac{1}{2}}, \qquad x_0 = 0.8,$$

to find the values of x_1, x_2 and x_3, giving your answers to 5 decimal places.

(3)

(*c*) Show that $\alpha = 0.80157$ is correct to 5 decimal places.

(3)

$R\sin(\theta \pm a)$

$R\cos(\theta \pm a)$

Trigonometry	I know and can use the expressions for $a\cos\theta + b\sin\theta$ in the forms $r\cos(\theta \pm a)$ and $r\sin(\theta \pm a)$	

1. (*a*) Using the identity $\cos (A + B) \equiv \cos A \cos B - \sin A \sin B$, prove that

$$\cos 2A \equiv 1 - 2 \sin^2 A.$$

(2)

(*b*) Show that

$$2 \sin 2\theta - 3 \cos 2\theta - 3 \sin \theta + 3 \equiv \sin \theta (4 \cos \theta + 6 \sin \theta - 3).$$

(4)

(*c*) Express $4 \cos \theta + 6 \sin \theta$ in the form $R \sin (\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(*d*) Hence, for $0 \leq \theta < \pi$, solve

$$2 \sin 2\theta = 3(\cos 2\theta + \sin \theta - 1),$$

giving your answers in radians to 3 significant figures, where appropriate.

(5)

2.

$$f(x) = 12 \cos x - 4 \sin x.$$

Given that $f(x) = R \cos (x + \alpha)$, where $R \geq 0$ and $0 \leq \alpha \leq 90°$,

(*a*) find the value of R and the value of α.

(4)

(*b*) Hence solve the equation

$$12 \cos x - 4 \sin x = 7$$

for $0 \leq x < 360°$, giving your answers to one decimal place.

(5)

(*c*) (i) Write down the minimum value of $12 \cos x - 4 \sin x$.

(1)

(ii) Find, to 2 decimal places, the smallest positive value of x for which this minimum value occurs.

(2)

3. **Figure 1**

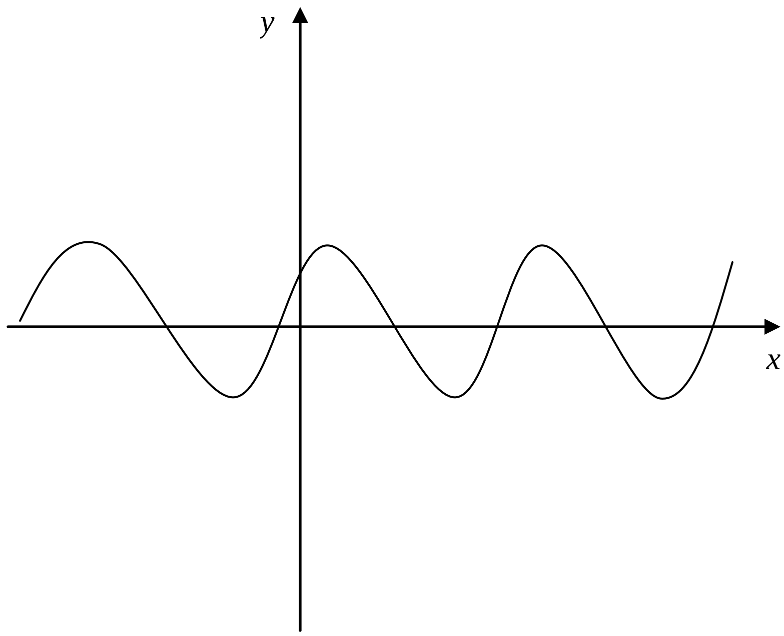


Figure 1 shows an oscilloscope screen.

The curve on the screen satisfies the equation $y = \sqrt{3}\cos x + \sin x$.

(*a*) Express the equation of the curve in the form $y = R\sin(x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(4)

(*b*) Find the values of x, $0 \leq x < 2\pi$, for which $y = 1$.

(4)

4. (*a*) Use the double angle formulae and the identity

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

to obtain an expression for cos 3*x* in terms of powers of cos *x* only.

(4)

(*b*) (i) Prove that

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} \equiv 2 \sec x, \qquad x \neq (2n+1)\frac{\pi}{2}.$$

(4)

(ii) Hence find, for $0 < x < 2\pi$, all the solutions of

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = 4.$$

(3)

5.

$$f(x) = 5 \cos x + 12 \sin x.$$

Given that $f(x) = R \cos (x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$,

(*a*) find the value of *R* and the value of *α* to 3 decimal places.

(4)

(*b*) Hence solve the equation

$$5 \cos x + 12 \sin x = 6$$

for $0 \leq x < 2\pi$.

(5)

(*c*) (i) Write down the maximum value of 5 cos *x* + 12 sin *x*.

(1)

(ii) Find the smallest positive value of *x* for which this maximum value occurs.

(2)

6. (*a*) Express 6 cos θ + 8 sin θ in the form R cos ($\theta - \alpha$), where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(*b*)
$$p(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \le \theta \le 2\pi.$$

Calculate

(i) the maximum value of p(θ),

(ii) the value of θ at which the maximum occurs.

(4)

7. (*a*) Express 2 cos 3x – 3 sin 3x in the form R cos (3x + α), where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your answers to 3 significant figures.

(4)

$$f(x) = e^{2x} \cos 3x.$$

(*b*) Show that f ′(x) can be written in the form

$$f'(x) = Re^{2x} \cos (3x + \alpha),$$

where R and α are the constants found in part (*a*).

(5)

(*c*) Hence, or otherwise, find the smallest positive value of x for which the curve with equation $y = f(x)$ has a turning point.

(3)

Coordinate Geometry and Transformations

Algebra and Functions	I know how to sketch the graph of $y = \|f(x)\|$.	
	I know how to sketch the graph of $y = f(\|x\|)$.	
	I can use combinations of transformations of $y=f(x)$ such as $af(x)$, $f(x)+a$, $f(x+a)$ and $y=f(ax)$.	
	I can identify x and y intercepts.	
	I can find normals and tangents.	

1. **Figure 1**

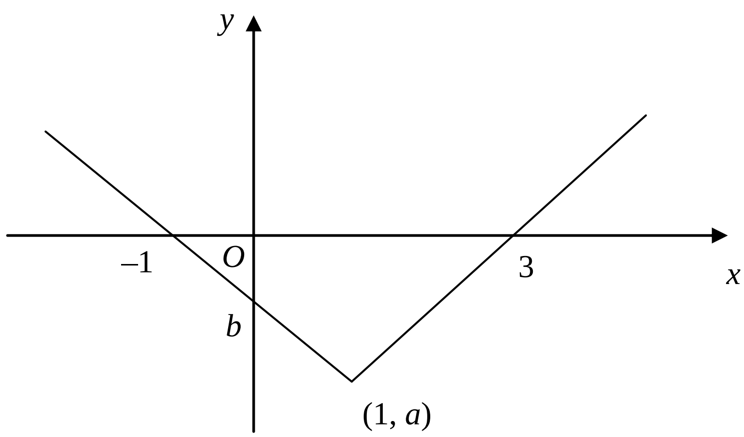


Figure 1 shows part of the graph of $y = f(x)$, $x \in \mathbb{R}$. The graph consists of two line segments that meet at the point $(1, a)$, $a < 0$. One line meets the x-axis at $(3, 0)$. The other line meets the x-axis at $(–1, 0)$ and the y-axis at $(0, b)$, $b < 0$.

In separate diagrams, sketch the graph with equation

(*a*) $y = f(x + 1)$,
(2)

(*b*) $y = f(|x|)$.
(3)

Indicate clearly on each sketch the coordinates of any points of intersection with the axes.

Given that $f(x) = |x – 1| – 2$, find

(*c*) the value of a and the value of b,
(2)

(*d*) the value of x for which $f(x) = 5x$.
(4)

2.

Figure 1

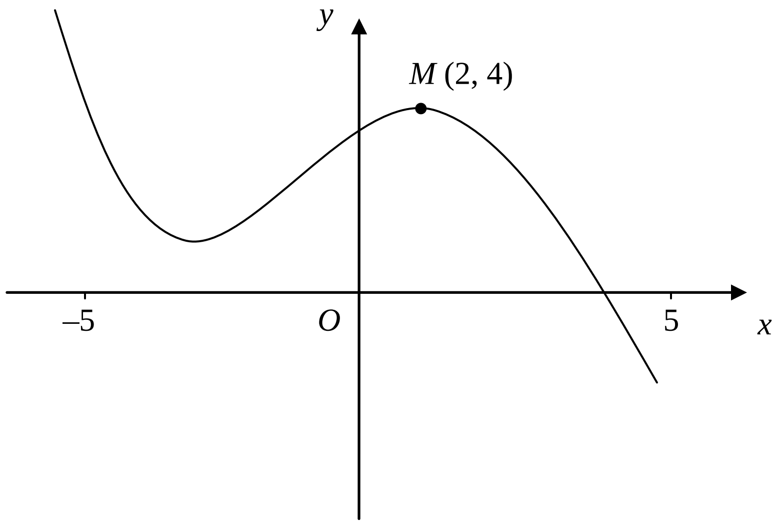


Figure 1 shows the graph of $y = \mathrm{f}(x)$, $-5 \leq x \leq 5$.

The point M (2, 4) is the maximum turning point of the graph.

Sketch, on separate diagrams, the graphs of

(*a*) $y = \mathrm{f}(x) + 3$,

(2)

(*b*) $y = |\mathrm{f}(x)|$,

(2)

(*c*) $y = \mathrm{f}(|x|)$.

(3)

Show on each graph the coordinates of any maximum turning points.

3.

Figure 1

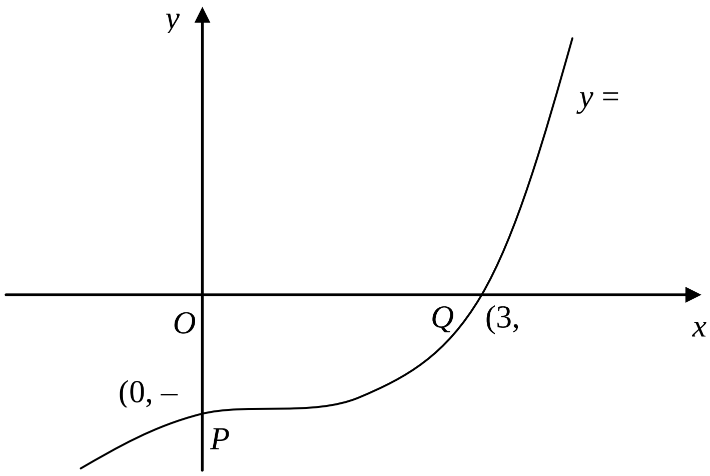


Figure 1 shows part of the curve with equation $y = \text{f}(x)$, $x \in \mathbb{R}$, where f is an increasing function of x. The curve passes through the points $P(0, –2)$ and $Q(3, 0)$ as shown.

In separate diagrams, sketch the curve with equation

(*a*) $y = |\text{f}(x)|$,

(3)

(*b*) $y = \text{f}^{-1}(x)$,

(3)

(*c*) $y = \frac{1}{2}\text{f}(3x)$.

(3)

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

4.

$$\text{f}(x) = x^4 - 4x - 8.$$

(*a*) Show that there is a root of $\text{f}(x) = 0$ in the interval $[-2, -1]$.

(3)

(*b*) Find the coordinates of the turning point on the graph of $y = \text{f}(x)$.

(3)

(*c*) Given that $\text{f}(x) = (x - 2)(x^3 + ax^2 + bx + c)$, find the values of the constants a, b and c.

(3)

(*d*) Sketch the graph of $y = \text{f}(x)$.

(3)

(*e*) Hence sketch the graph of $y = |\text{f}(x)|$.

(1)

5. For the constant k, where $k > 1$, the functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln(x + k), \quad x > -k,$$

$$\mathrm{g}: x \mapsto |2x - k|, \quad x \in \mathbb{R}.$$

(*a*) On separate axes, sketch the graph of f and the graph of g.

On each sketch state, in terms of k, the coordinates of points where the graph meets the coordinate axes.

(5)

(*b*) Write down the range of f.

(1)

(*c*) Find $\mathrm{fg}\left(\frac{k}{4}\right)$ in terms of k, giving your answer in its simplest form.

(2)

The curve C has equation $y = \mathrm{f}(x)$. The tangent to C at the point with x-coordinate 3 is parallel to the line with equation $9y = 2x + 1$.

(*d*) Find the value of k.

(4)

6. A curve C has equation

$$y = 3\sin 2x + 4\cos 2x, \quad -\pi \leq x \leq \pi .$$

The point $A(0, 4)$ lies on C.

(*a*) Find an equation of the normal to the curve C at A.

(5)

(*b*) Express y in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 significant figures.

(4)

(*c*) Find the coordinates of the points of intersection of the curve C with the x-axis.
Give your answers to 2 decimal places.

(4)

7.

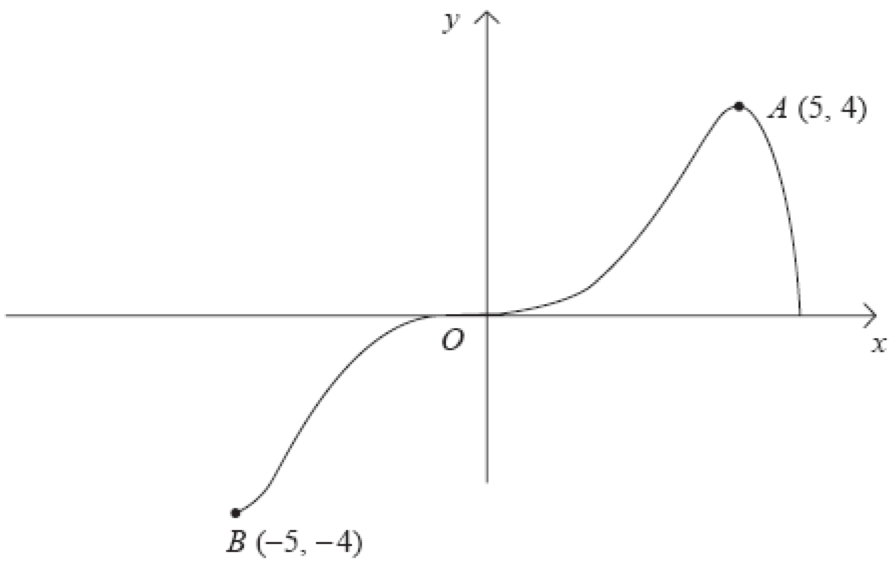


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$.

The curve passes through the origin O and the points $A(5, 4)$ and $B(-5, -4)$.

In separate diagrams, sketch the graph with equation

(*a*) $y = |f(x)|$,

(3)

(*b*) $y = f(|x|)$,

(3)

(*c*) $y = 2f(x + 1)$.

(4)

On each sketch, show the coordinates of the points corresponding to A and B.

8. The point P lies on the curve with equation

$$y = 4e^{2x+1}.$$

The y-coordinate of P is 8.

(*a*) Find, in terms of ln 2, the x-coordinate of P.

(2)

(*b*) Find the equation of the tangent to the curve at the point P in the form $y = ax + b$, where a and b are exact constants to be found.

(4)

9.

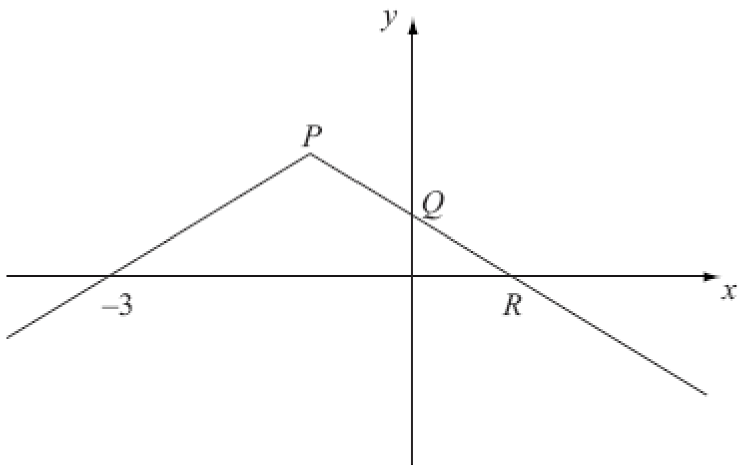


Figure 1

Figure 1 shows the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$,

The graph consists of two line segments that meet at the point P.

The graph cuts the y-axis at the point Q and the x-axis at the points (–3, 0) and R.

Sketch, on separate diagrams, the graphs of

(*a*) $y = |\mathrm{f}(x)|$, **(2)**

(*b*) $y = \mathrm{f}(-x)$. **(2)**

Given that $\mathrm{f}(x) = 2 - |x + 1|$,

(*c*) find the coordinates of the points P, Q and R, **(3)**

(*d*) solve $\mathrm{f}(x) = \frac{1}{2}x$. **(5)**

10.

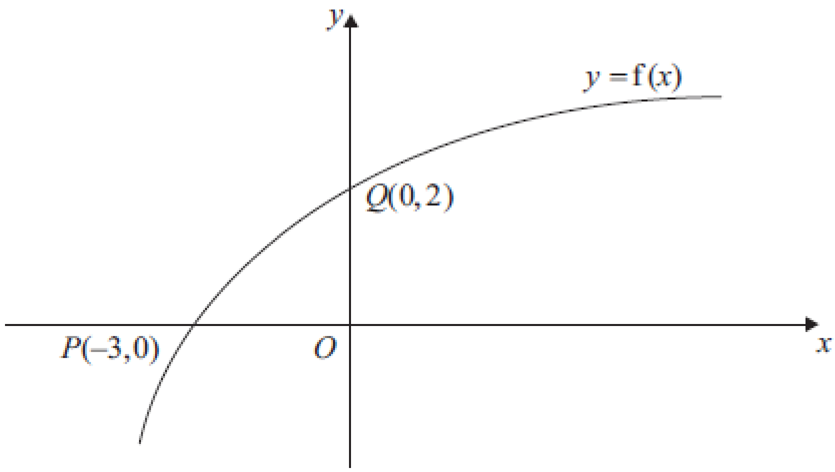


Figure 1

Figure 1 shows part of the curve with equation $y = \text{f}(x)$, $x \in \mathbb{R}$.

The curve passes through the points $Q(0, 2)$ and $P(-3, 0)$ as shown.

(*a*) Find the value of ff (−3).

(2)

On separate diagrams, sketch the curve with equation

(*b*) $y = \text{f}^{-1}(x)$,

(2)

(*c*) $y = \text{f}(|x|) - 2$,

(2)

(*d*) $y = 2\text{f}\left(\frac{1}{2}x\right)$.

(3)

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

11.

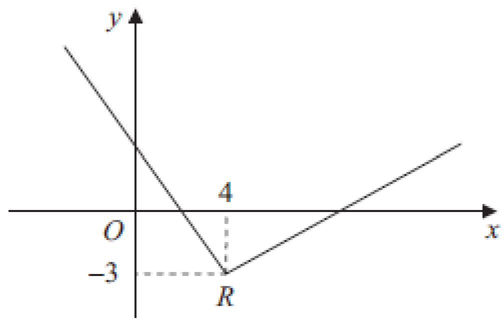


Figure 1

Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point R (4, – 3), as shown in Figure 1.

Sketch, on separate diagrams, the graphs of

(*a*) $y = 2\mathrm{f}(x + 4)$,

(3)

(*b*) $y = |\mathrm{f}(-x)|$.

(3)

On each diagram, show the coordinates of the point corresponding to R.

Exponential Equations

Exponentials and Logarithms	I know the function e^x and its graph.	
	I understand how the graph of e^x may be transformed, eg: $e^{ax + b} + c$.	
	I understand the function ln(x) and its graph.	
	I understand the relationship between ln(x) and e^x.	
	I can solve equations of the form $e^{ax + b} = p$.	
	I can solve equations of the form $\ln(ax + b) = q$.	

1. A particular species of orchid is being studied. The population p at time t years after the study started is assumed to be

$$p = \frac{2800ae^{0.2t}}{1+ae^{0.2t}}, \text{ where } a \text{ is a constant.}$$

Given that there were 300 orchids when the study started,

(*a*) show that $a = 0.12$,
(3)

(*b*) use the equation with $a = 0.12$ to predict the number of years before the population of orchids reaches 1850.
(4)

(*c*) Show that $p = \dfrac{336}{0.12 + e^{-0.2t}}$.
(1)

(*d*) Hence show that the population cannot exceed 2800.
(2)

2. A heated metal ball is dropped into a liquid. As the ball cools, its temperature, T °C, t minutes after it enters the liquid, is given by

$$T = 400e^{-0.05t} + 25, \quad t \geq 0.$$

(*a*) Find the temperature of the ball as it enters the liquid.
(1)

(*b*) Find the value of t for which $T = 300$, giving your answer to 3 significant figures.
(4)

(*c*) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. Give your answer in °C per minute to 3 significant figures.
(3)

(*d*) From the equation for temperature T in terms of t, given above, explain why the temperature of the ball can never fall to 20 °C.
(1)

3. The radioactive decay of a substance is given by

$$R = 1000e^{-ct}, \quad t \geq 0.$$

where R is the number of atoms at time t years and c is a positive constant.

(*a*) Find the number of atoms when the substance started to decay.

(1)

It takes 5730 years for half of the substance to decay.

(*b*) Find the value of c to 3 significant figures.

(4)

(*c*) Calculate the number of atoms that will be left when $t = 22\,920$.

(2)

(*d*) Sketch the graph of R against t .

(2)

4. The value of Bob's car can be calculated from the formula

$$V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500.$$

where V is the value of the car in pounds (£) and t is the age in years.

(*a*) Find the value of the car when $t = 0$.

(1)

(*b*) Calculate the exact value of t when $V = 9500$.

(4)

(*c*) Find the rate at which the value of the car is decreasing at the instant when $t = 8$.
Give your answer in pounds per year to the nearest pound.

(4)

5. The mass, m grams, of a leaf t days after it has been picked from a tree is given by

$$m = pe^{-kt},$$

where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(*a*) Write down the value of p.

(1)

(*b*) Show that $k = \frac{1}{4}\ln 3$.

(4)

(*c*) Find the value of t when $\frac{dm}{dt} = -0.6 \ln 3$.

(6)