Core Maths C2

Revision Notes

November 2012

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1 Algebra

Polynomials: +, –, ×, ÷.

A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$$

where all the powers of the variable, x, are positive integers or 0.

+, –, × of polynomials are easy, ÷ must be done by long division.

Factorising

General examples of factorising:

$$2ab + 6ac^2 = 2a(b + 3c^2)$$

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

$$x^2 - 6x = x(x - 6)$$

$$6x^2 - 11x - 10 = (3x + 2)(2x - 5)$$

Standard results:

$$x^2 - y^2 = (x - y)(x + y), \qquad \text{difference of two squares}$$

$$(x + y)^2 = x^2 + 2xy + y^2,$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

Long division

See examples in book

Remainder theorem

If 627 is divided by 6 the quotient is 104 and the remainder is 3.
This can be written as $627 = 6 \times 104 + 3$.

In the same way, if a polynomial

$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$ is divided by $(cx + d)$ to give a quotient, $Q(x)$ with a remainder r, then r will be a constant (since the divisor is of degree one) and we can write

$$P(x) = (cx + d) \times Q(x) + r$$

If we now choose the value of x which makes $(cx + d) = 0 \Rightarrow x = {}^{-d}/_{c}$

then we have $P\left({}^{-d}/_{c}\right) = 0 \times Q(x) + r$

$$\Rightarrow \qquad P\left({}^{-d}/_{c}\right) = r.$$

Theorem: If we substitute $x = {}^{-d}/_{c}$ in the polynomial we obtain the remainder that we would have after dividing the polynomial by $(cx + d)$.

Example: The remainder when $P(x) = 4x^3 - 2x^2 + ax - 4$ is divided by $(2x - 3)$ is 7. Find a.

Solution: Choose the value of x which makes $(2x - 3) = 0$, i.e. $x = {}^{3}/_{2}$.

Then the remainder is $P\left(\tfrac{3}{2}\right) = 4 \times \left(\tfrac{3}{2}\right)^3 - 2 \times \left(\tfrac{3}{2}\right)^2 + a \times \left(\tfrac{3}{2}\right) - 4 = 7$

$\Rightarrow \quad \tfrac{27}{2} - \tfrac{9}{2} + a \times \left(\tfrac{3}{2}\right) - 4 = 7 \quad \Rightarrow \quad 5 + a \times \left(\tfrac{3}{2}\right) = 7$

$\Rightarrow \quad a = \tfrac{4}{3}$.

You may be given a polynomial with two unknown letters, a and b. You will also be given two pieces of information to let you form two simultaneous equations in a and b.

Factor theorem

Theorem: If, in the remainder theorem, $r = 0$ then $(cx + d)$ is a factor of $P(x)$

$\Rightarrow \quad P\left(\tfrac{-d}{c}\right) = 0 \quad \Leftrightarrow \quad (cx + d)$ is a factor of $P(x)$.

Example: A quadratic equation has solutions (roots) $x = {}^{-1}/_{2}$ and $x = 3$. Find the quadratic equation in the form $ax^2 + bx + c = 0$

Solution: The equation has roots $x = {}^{-}\tfrac{1}{2}$ and $x = 3$

$\Rightarrow$ it must have factors $(2x + 1)$ and $(x - 3)$ by the factor theorem

$\Rightarrow$ the equation is $(2x + 1)(x - 3) = 0$

$\Rightarrow \quad 2x^2 - 5x - 3 = 0$.

Example: Show that $(x - 2)$ is a factor of $P(x) = 6x^3 - 19x^2 + 11x + 6$ and hence factorise the expression completely.

Solution: Choose the value of x which makes $(x - 2) = 0$, i.e. $x = 2$

$\Rightarrow$ remainder $= P(2) = 6 \times 8 - 19 \times 4 + 11 \times 2 + 6 = 48 - 76 + 22 + 6 = 0$

$\Rightarrow$ $(x - 2)$ is a factor by the factor theorem.

We have a cubic and so we can see that the other factor must be a quadratic of the form

$(6x^2 + ax - 3)$ and we can write

$6x^3 - 19x^2 + 11x + 6 = (x - 2)(6x^2 + ax - 3)$

Multiplying out we see that the $6x^3$ and $+6$ terms are correct.

The x term is $11x$

Multiplying out the x term comes from $x \times -3 + -2 \times ax$ which must come to $11x$

$\Rightarrow \quad a = -7$.

We must now check the x^2 term which is $-19x^2$.

Multiplying out with $a = -7$

the x^2 term is $x \times (-7x) + (-2) \times 6x^2 = -7x^2 - 12x^2 = -19x^2$, which works!.

$$\Rightarrow \quad 6x^3 - 19x^2 + 11x + 6 = (x - 2)(6x^2 - 7x - 3)$$

$$= (x - 2)(2x - 3)(3x + 1)$$

which is now factorised completely.

Choosing a suitable factor

To choose a suitable factor we look at the coefficient of the highest power of x and the constant (the term without an x).

Example: Factorise $2x^3 + x^2 - 13x + 6$.

Solution: 2 is the coefficient of x^3 and 2 has factors of 2 and 1.

6 is the constant and 6 has factors of 1, 2, 3 and 6

so the possible linear factors of $2x^3 + x^2 - 13x + 6$ are

$(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $(x \pm 6)$

$(2x \pm 1)$, $(2x \pm 2)$, $(2x \pm 3)$, $(2x \pm 6)$

But $(2x \pm 2) = 2(x \pm 1)$ and $(2x \pm 6) = 2(x \pm 3)$, so they are not new factors.

We now test the possible factors using the factor theorem until we find one that works.

Test $(x - 1)$, put $x = 1$ giving $2 \times 1^3 + 1^2 - 13 \times 1 + 6 \neq 0$

Test $(x + 1)$, put $x = -1$ giving $2 \times (-1)^3 + (-1)^2 - 13 \times (-1) + 6 \neq 0$

Test $(x - 2)$, put $x = 2$ giving $2 \times 2^3 + 2^2 - 13 \times 2 + 6 = 16 + 4 - 26 + 6 = 0$

and since the result is zero $(x - 2)$ is a factor.

We now divide in, as in the previous example, to give

$$2x^3 + x^2 - 13x + 6 = (x - 2)(2x^2 + 5x - 3)$$

$$= (x - 2)(2x - 1)(x + 3).$$

Cubic equations

Factorise using the factor theorem then solve.

N.B. The quadratic factor might not factorise in which case you will need to use the formula for this part.

Example: Solve the equation $x^3 - x^2 - 3x + 2 = 0$.

Solution: Possible factors are $(x \pm 1)$ and $(x \pm 2)$.

Put $x = 1$ we have $1^3 - 1^2 - 3\times1 + 2 = -1 \neq 0$

$\Rightarrow$ $(x - 1)$ is **not** a factor

Putting $x = 2$ we have $2^3 - 2^2 - 3\times2 + 2 = 8 - 4 - 6 + 2 = 0$

$\Rightarrow$ $(x - 2)$ is a factor

$\Rightarrow$ $x^3 - x^2 - 3x + 2 = (x - 2)(x^2 + x - 1) = 0$

$\Rightarrow$ $x = 2$ or $x^2 + x - 1 = 0$ – this will not factorise so we use the formula

$$\Rightarrow \quad x = 2 \text{ or } x = \frac{-1 \pm \sqrt{(-1)^2 - -4\times1\times1}}{2\times1} = 0.618 \text{ or } -1.31$$

2 Trigonometry

Radians

A radian is the angle subtended at the centre of a circle by an arc of length equal to the radius.

Connection between radians and degrees

$180^o = \pi^c$

Degrees	30	45	60	90	120	135	150	180	270	360
Radians	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	$3\pi/2$	2π

Arc length , area of a sector and area of a segment

Arc length $s = r\theta$ and area of sector $A = \frac{1}{2}r^2\theta$.

Area of segment = area sector – area of triangle = $\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$.

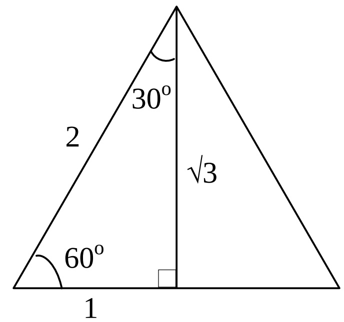
Trig functions

Basic results

$\tan A = \dfrac{\sin A}{\cos A}$; $\sin(-A) = -\sin A$; $\cos(-A) = \cos A$; $\tan(-A) = -\tan A$;

Exact values for 30°, 45° and 60°

From the equilateral triangle of side 2 we can read off

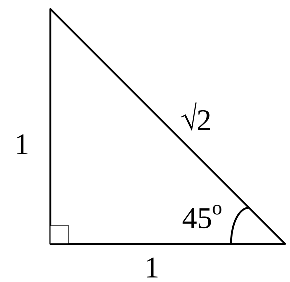


$\sin 60^\circ = \sqrt{3}/2$ $\qquad$ $\sin 30^0 = \frac{1}{2}$

$\cos 60^\circ = \frac{1}{2}$ $\qquad$ $\cos 30^\circ = \sqrt{3}/2$

$\tan 60^\circ = \sqrt{3}$ $\qquad$ $\tan 30^\circ = 1/\sqrt{3}$

and from the isosceles right–angled triangle with sides 1, 1, $\sqrt{2}$ we can read off

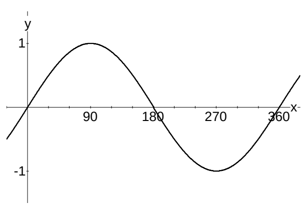


$\sin 45^0 = 1/\sqrt{2}$

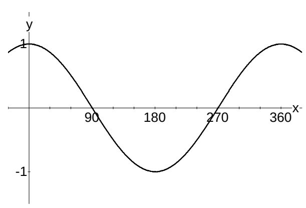
$\cos 45^\circ = 1/\sqrt{2}$

$\tan 45^\circ = 1$

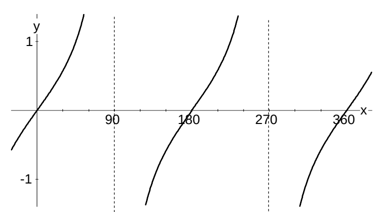
Graphs of trig functions



$y = \sin x$



$y = \cos x$



$y = \tan x$

Graphs of $y = \sin nx$, $y = \sin(-x)$, $y = \sin(x + n)$ etc.

You should know the shapes of these graphs

$y = \sin \mathbf{3}x$ is like $y = \sin x$ but repeats itself **3** times between 0° and 360°

$y = \sin(-x) = -\sin x$ and $y = \tan(-x) = -\tan x$ are the graphs of $y = \sin x$ and $y = \tan x$ reflected in the x-axis.

$y = \cos(-x) = \cos x$ is just the graph of $y = \cos x$.

$y = \sin(x + 30)$ is the graph of $y = \sin x$ translated through $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$.

Sine & Cosine rules and area of triangle

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$: **be careful** – the sine rule always gives you two answers for each angle so if possible do not use the sine rule to find the largest angle as it might be obtuse;

or you might want both answers if it is possible to draw two different triangles from the information given.

$a^2 = b^2 + c^2 - 2bc\cos A$ etc. Unique answers here!

Area of a triangle $= \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$.

Identities

$$\mathbf{\tan A = \frac{\sin A}{\cos A}}$$

Example: Solve $3\sin x = 4\cos x$.

Solution: First divide both sides by $\cos x$

$\Rightarrow \quad 3\frac{\sin x}{\cos x} = 4 \quad \Rightarrow \quad 3\tan x = 4 \quad \Rightarrow \quad \tan x = {}^4/_3$

$\Rightarrow \quad x = 53.1^\circ$, or 233.1°.

$$\mathbf{\sin^2 A + \cos^2 A = 1}$$

Example: Given that $\cos A = {}^5/_{13}$ and that $270^\circ < A < 360^\circ$, find $\sin A$ and $\tan A$.

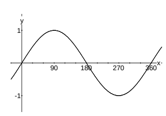
Solution: We know that $\sin^2 A + \cos^2 A = 1$

$\Rightarrow \quad \sin^2 A = 1 - \cos^2 A$

$\Rightarrow \quad \sin^2 A = 1 - ({}^5/_{13})^2$

$\Rightarrow \quad \sin A = \pm{}^{12}/_{13.}$

But $270^\circ < A < 360^\circ$



$\Rightarrow \quad \sin A$ is negative

$\Rightarrow \quad \sin A = -{}^{12}/_{13}$.

Also $\tan A = \frac{\sin A}{\cos A}$

$\Rightarrow \quad \tan A = \frac{-12/13}{5/13} = -{}^{12}/_5 = -2.4$.

Example: Solve $2\sin^2 x + \sin x - \cos^2 x = 1$

Solution: Rewriting $\cos^2 x$ in terms of $\sin x$ will make life easier

so using $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad \cos^2 x = 1 - \sin^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - \cos^2 x = 1$

$\Rightarrow \quad 2\sin^2 x + \sin x - (1 - \sin^2 x) = 1$

$\Rightarrow \quad 3\sin^2 x + \sin x - 2 = 0$

$\Rightarrow \quad (3\sin x - 2)(\sin x + 1) = 0$

$\Rightarrow \quad \sin x = {}^2/_3$ or -1

$\Rightarrow \quad x = 41.8^o,\ 138.9^o$, or 270^o.

N.B. If asked to give answers in radians, you are allowed to work in degrees as above and then convert to radians by multiplying by $^{\pi}/_{180}$

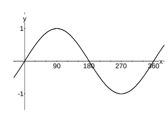
So answers in radians would be

$x = 41.8103 \times {}^{\pi}/_{180} = 0.730$, or $138.1897 \times {}^{\pi}/_{180} = 2.41$, or $270 \times {}^{\pi}/_{180} = {}^{3\pi}/_{2}$.

Trigonometric equations

Example: Solve $\sin(x - {}^{\pi}/_4) = 0.5$ for $0^c \le x \le 2\pi^c$, giving your answers in radians in terms of π.

Solution: First we know that $\sin 60^o = 0.5$, and $60^o = {}^{\pi}/_3$ radians



$\Rightarrow \quad x - {}^{\pi}/_4 = {}^{\pi}/_3$ or $\pi - {}^{\pi}/_3 = {}^{2\pi}/_3 \Rightarrow \quad x = {}^{7\pi}/_{12}$ or ${}^{11\pi}/_{12}$.

Example: Solve $\sin 2x = 0.471$ for $0^o \le x \le 360^o$, giving your answers to the nearest degree.

Solution: First put $X = 2x$ and find **all** solutions of $\sin X = 0.471$ for $0^o \le X \le 720^o$

$\Rightarrow \quad X = 28.1$, or $180 - 28.1 = 151.9$

or $28.1 + 360 = 388.1$, or $151.9 + 360 = 511.9$

i.e. $X = 28.1, 151.9, 388.1, 511.9$

$\Rightarrow \quad x = 14^o, 76^o, 194^o, 256^o$ to the nearest degree.

There are several examples in the book.

3 Coordinate Geometry

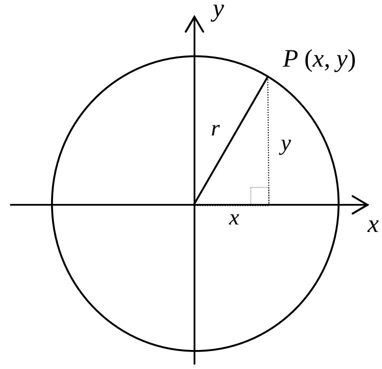
Mid point

The mid point of the line joining $P\ (a_1, b_1)$ and $Q\ (a_2, b_2)$ is $\left(\frac{1}{2}(a_1 + a_2),\ \frac{1}{2}(b_1 + b_2)\right)$.

Circle

Centre at the origin

Take any point, P, on a circle centre the origin and radius 5.



Suppose that P has coordinates (x, y)

Using Pythagoras' Theorem we have

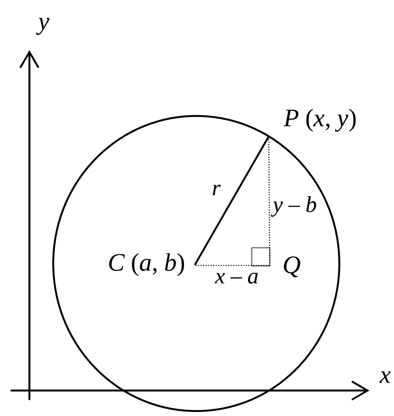
$$x^2 + y^2 = 5^2 \quad \Rightarrow \quad x^2 + y^2 = 25$$

which is the equation of the circle.

and in general the equation of a circle centre (0, 0) and radius r is

$$x^2 + y^2 = r^2.$$

General equation



In the circle shown the centre is C, (a, b), and the radius is r.

$CQ = x - a$ and $PQ = y - b$

and, using Pythagoras

$\Rightarrow \quad CQ^2 + PQ^2 = r^2$

$\Rightarrow \quad (x - a)^2 + (y - b)^2 = r^2,$

which is the general equation of a circle.

Example: Find the centre and radius of the circle whose equation is

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

Solution: First complete the square in both x and y to give

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 12 + 4 + 9 = 25$$

$$\Rightarrow \quad (x - 2)^2 + (y + 3)^2 = 5^2$$

which is the equation of a circle with centre (2, –3) and radius 5.

Example: Find the equation of the circle on the line joining A, (3, 5), and B, (8, –7), as diameter.

Solution: The centre is the mid point of AB is $\left(\tfrac{1}{2}(3+8),\ \tfrac{1}{2}(5-7)\right) = (5½, -1)$
and the radius is $½\, AB = ½\sqrt{(8-3)^2+(-7-5)^2} = 6.5$

$\Rightarrow$ equation is $(x - 5.5)^2 + (y + 1)^2 = 6.5^2$.

Equation of tangent

Example: Find the equation of the tangent to the circle $x^2 + 2x + y^2 - 4y = 164$ which passes through the point of the circle (–6, 14).

Solution: First complete the square in x and in y to give

$(x + 1)^2 + (y - 2)^2 = 169$.

Next find the gradient of the radius from the centre (–1, 2) to the point (– 6, 14) which is $^{12}/_{-5}$

$\Rightarrow$ gradient of the tangent at that point is $^{5}/_{12}$, since the tangent is perpendicular to the radius and product of gradients of perpendicular lines is –1

$\Rightarrow$ equation of the tangent is $y - 14 = \dfrac{5}{12}(x + 6)$

$\Rightarrow$ $12y - 5x = 198$.

Example: Find the intersection of the line $y = 2x + 4$ with the circle $x^2 + y^2 = 5$.

Solution: Put $y = 2x + 4$ in $x^2 + y^2 = 5$ to give $x^2 + (2x + 4)^2 = 5$

$\Rightarrow$ $x^2 + 4x^2 + 16x + 16 = 5$

$\Rightarrow$ $5x^2 + 16x + 11 = 0$

$\Rightarrow$ $(5x + 11)(x + 1) = 0$

$\Rightarrow$ $x = -2.2$ or -1

$\Rightarrow$ $y = -0.4$ or 2

$\Rightarrow$ line intersects circle at $(-2.2, -0.4)$ and $(-1, 2)$

If the two points of intersection are the same point then the line is a *tangent*.

Note. You should know that the angle in a semi-circle is a right angle and that the perpendicular from the centre to a chord bisects the chord (cuts it exactly in half) .

4 Sequences and series

Geometric series

Finite geometric series

A *geometric series* is a series in which each term is a constant amount times the previous term: this *constant amount* is called the *common ratio.*

The common ratio can be ≥ 1 *or* ≤ 1, and positive *or* negative.

Examples: 2, 6, 18, 54, 162, 486, ... with common ratio 3,

40, 20, 10, 5, 2½, 1¼, ... with common ratio ½,

½, –2, 8, –32, 128, –512, ... with common ratio –4.

Generally a geometric series can be written as

$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots$ up to n terms

where a is the first term and r is the common ratio.

The nth term is $u_n = ar^{n-1}$.

The sum of the first n terms of the above geometric series is

$$S_n = a\frac{1-r^n}{1-r} = a\frac{r^n-1}{r-1}.$$

Example: Find the n^{th} term and the sum of the first 11 terms of the geometric series whose 3rd term is 2 and whose 6th term is –16.

Solution: $x_6 = x_3 \times r^3 \Rightarrow -16 = 2 \times r^3 \Rightarrow r^3 = -8$

$\Rightarrow r = -2$

Now $x_3 = x_1 \times r^2 \quad x_1 = x_3 \div r^2 = 2 \div (-2)^2$

$\Rightarrow x_1 = ½$

$\Rightarrow n^{th}$ term, $x_n = ar^{n-1} = ½ \times (-2)^{n-1}$

and the sum of the first 11 terms is

$$S_{11} = \frac{1}{2}\,\frac{(-2)^{11}-1}{-2-1} = \frac{-2049}{-6} \quad \Rightarrow \quad S_{11} = 341\tfrac{1}{2}$$

Infinite geometric series

When the common ratio is between –1 and +1 the series converges to a limit.

$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots$ up to n terms

and $S_n = a\dfrac{1-r^n}{1-r}$.

Since $|r| < 1$, $r^n \to 0$ as $n \to \infty$ and so

$$S_n \to S_\infty = \frac{a}{1-r}$$

Example: Show that the following geometric series converges to a limit and find its sum to infinity. $S = 16 + 12 + 9 + 6\tfrac{3}{4} + \ldots$

Solution: Firstly the common ratio is $^{12}/_{16} = \tfrac{3}{4}$ which lies between -1 and +1 therefore the sum converges to a limit.

The sum to infinity $S_\infty = \dfrac{a}{1-r} = \dfrac{16}{1-\frac{3}{4}}$

$\Rightarrow \quad S_\infty = 64$

Proof of the formula for the sum of a geometric series

You **must** know this proof.

$$S_n = a + ar + ar^2 + ar^3 + \ldots\ ar^{n-2} + ar^{n-1}, \qquad \text{multiply through by } r$$

$$\Rightarrow \quad r \times S_n = \quad ar + ar^2 + ar^3 + \ldots\ ar^{n-2} + ar^{n-1} + ar^n \qquad \text{subtract}$$

$$\Rightarrow S_n - r \times S_n = a + 0 + 0 + 0 + \ldots\ 0 + 0 - ar^n$$

$$\Rightarrow \quad (1-r)\,S_n = a - ar^n = a(1-r^n)$$

$$\Rightarrow \quad S_n = a\frac{1-r^n}{1-r} = a\frac{r^n-1}{r-1}.$$

Notice that if $-1 < r < +1$ then $r^n \to 0$ and

$$S_n \to S_\infty = \frac{a}{1-r}.$$

Binomial series for positive integral index

Pascal's triangle

When using Pascal's triangle we think of the top row as **row 0.**

row 0							1						
row 1						1		1					
row 2					1		2		1				
row 3				1		3		3		1			
row 4			1		4		6		4		1		
row 5		1		5		10		10		5		1	
row 6	1		6		15		20		15		6		1

To expand $(a + b)^{\mathbf{6}}$ we first write out all the terms of 'degree **6**' in order of decreasing powers of a to give

$$a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$$

and then fill in the coefficients using row **6** of the triangle to give

$$\mathbf{1}a^6 + \mathbf{6}a^5b + \mathbf{15}a^4b^2 + \mathbf{20}a^3b^3 + \mathbf{15}a^2b^4 + \mathbf{6}ab^5 + \mathbf{1}b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Factorials

Factorial n, written as $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

So $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Binomial coefficients or ^nC_r or $\binom{n}{r}$

If we think of row 6 starting with the 0^{th} term we use the following notation

0^{th} term	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term
1	6	15	20	15	6	1
$^6C_0$	$^6C_1$	$^6C_2$	$^6C_3$	$^6C_4$	$^6C_5$	$^6C_6$
$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$

where the *binomial coefficients* ^nC_r or $\binom{n}{r}$ are defined by

$$ {}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)! \times r!} $$

This is particularly useful for calculating the numbers further down in Pascal's triangle

e.g. The fourth term in row 15 is

$$ {}^{15}C_4 \ \binom{15}{4} = \frac{15!}{(15-4)!\,4!} = \frac{15!}{11!\,4!} = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 15 \times 7 \times 13 = 1365 $$

You may find an ${}^nC_r$ button on your calculator.

Example: Find the coefficient of x^3 in the expansion of $(1 - 2x)^5$.

Solution: The term in x^3 is ${}^5C_3 \times 1^2 \times (-2x)^3 = 10 \times (-8)x^3 = -80x^3$

so the coefficient of x^3 is -80.

5 Exponentials and logarithms

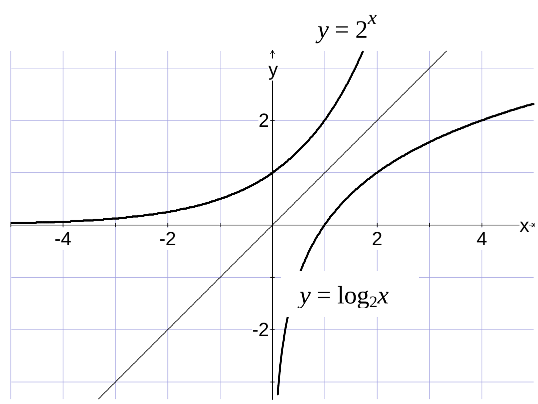
Graphs of exponentials and logarithms

$y = 2^x$ is an *exponential* function

and its inverse is the *logarithm* function

$y = \log_2 x$.

Remember that the graph of an inverse function is the reflection of the original graph in $y = x$.



Rules of logarithms

$\log_a x = y \iff x = a^y$

$\log_a xy = \log_a x + \log_a y$

$\log_a (x \div y) = \log_a x - \log_a y$

$\log_a x^n = n \log_a x$

$\log_a 1 = 0$

$\log_a a = 1$

Note: $\log_{10} x$ is often written as $\lg x$

Example: Find $\log_3 81$.

Solution: Write $\log_3 81 = y$

$\Rightarrow \quad 81 = 3^y \quad \Rightarrow \quad y = 4 \quad \Rightarrow \quad \log_3 81 = 4.$

To solve 'log' equations we can either use the rules of logarithms to end with

$\log_a$ ✷ $= \log_a$ ✸ $\quad \Rightarrow \quad$ ✷ $=$ ✸

or $\log_a$ ✷ $=$ ✸ $\quad \Rightarrow \quad$ ✷ $= a^{✸}$

Example: Solve $\log_a 40 - 3\log_a x = \log_a 5$

Solution: $\log_a 40 - 3\log_a x = \log_a 5$

$\Rightarrow \quad \log_a 40 - \log_a 5 = 3\log_a x$

$\Rightarrow \quad \log_a (40 \div 5) = \log_a x^3$

$\Rightarrow \quad \log_a 8 = \log_a x^3 \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2.$

Example: Solve $\log_5 x^2 = 3 + ½\log_5 x$.

Solution: $\log_5 x^2 = 3 + ½\log_5 x$

$\Rightarrow \quad 2\log_5 x - ½\log_5 x = 3$

$\Rightarrow \quad 1.5\log_5 x = 3 \quad \Rightarrow \quad \log_5 x = 2$

$\Rightarrow \quad x = 5^2 = 25.$

Changing the base of a logarithm

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example: Find $\log_4 29$.

Solution: $\log_4 29 = \dfrac{\log_{10} 29}{\log_{10} 4} = \dfrac{1.4624}{0.6021} = 2.43.$

A particular case

$\log_a b = \dfrac{\log_b b}{\log_b a} = \dfrac{1}{\log_b a}$ This gives a source of exam questions.

Example: Solve $\log_4 x - 6\log_x 4 = 1$

Solution: $\Rightarrow \quad \log_4 x - \dfrac{6}{\log_4 x} = 1 \quad \Rightarrow \quad (\log_4 x)^2 - \log_4 x - 6 = 0$

$\Rightarrow \quad (\log_4 x - 3)(\log_4 x + 2) = 0$

$\Rightarrow \quad \log_4 x = 3 \text{ or } -2$

$\Rightarrow \quad x = 4^3 \text{ or } 4^{-2} \quad \Rightarrow \quad x = 64 \text{ or } {}^1/_{16}.$

Equations of the form $a^x = b$

Example: Solve $5^x = 13$

Solution: Take logs of both sides

$\Rightarrow$ $\log_{10} 5^x = \log_{10} 13$

$\Rightarrow$ $x \log_{10} 5 = \log_{10} 13$

$$\Rightarrow \quad x = \frac{\log_{10} 13}{\log_{10} 5} = \frac{1.1139}{0.6990} = 1.59.$$

6 Differentiation

Increasing and decreasing functions

y is an increasing function if its gradient is positive, $\frac{dy}{dx} > 0$;

y is an increasing function if its gradient is negative, $\frac{dy}{dx} < 0$

Example: For what values of x is $y = x^3 - x^2 - x + 7$ an increasing function.

Solution: $y = x^3 - x^2 - x + 7$

$\Rightarrow$ $\frac{dy}{dx} = 3x^2 - 2x - 1$

For an increasing function we want values of x for which $3x^2 - 2x - 1 > 0$

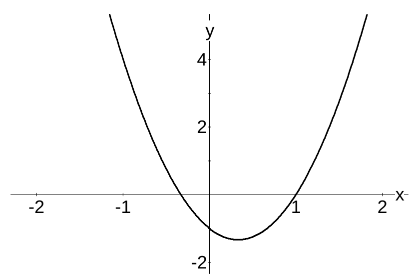
Find solutions of $3x^2 - 2x - 1 = 0$

$\Rightarrow$ $(3x + 1)(x - 1) = 0$

$\Rightarrow$ $x = {}^{-1}/_3$ or 1

so graph of $3x^2 - 2x - 1$ meets x-axis at ${}^{-1}/_3$ and 1 and is above x-axis for

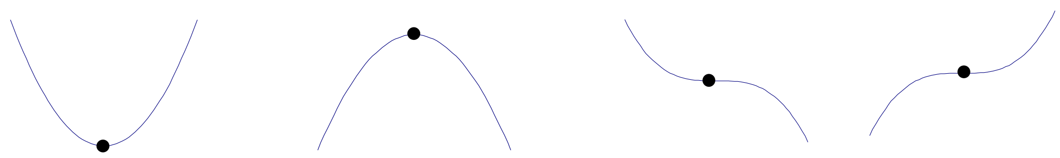
$x < {}^{-1}/_3$ or $x > 1$



So $y = x^3 - x^2 - x + 7$ is an increasing function for $x < {}^{-1}/_3$ or $x > 1$.

Stationary points and local maxima and minima (turning points).

Any point where the gradient is zero is called a *stationary point*.



minimum *maximum*

Turning points Stationary points of inflection

Local maxima and minima are called *turning points.*

The gradient at a local maximum or minimum is 0.

Therefore to find *max* and *min*

firstly – differentiate and find the values of x which give gradient, $\frac{dy}{dx}$, equal to zero:

secondly – find second derivative $\frac{d^2y}{dx^2}$ and substitute value of x found above –

second derivative positive $\Rightarrow$ minimum, and
second derivative negative $\Rightarrow$ maximum:

N.B. If $\frac{d^2y}{dx^2} = 0$, it does not help! In this case you will need to **find the gradient just before and just after** the value of x. *Be careful:* you might have a stationary point of inflection

thirdly – substitute x to find the value of y and give both coordinates in your answer.

Using second derivative

Example:

Find the local maxima and minima of the curve with equation $y = x^4 + 4x^3 - 8x^2 - 7$.

Solution:

$y = x^4 + 4x^3 - 8x^2 - 7$.

First find $\frac{dy}{dx} = 4x^3 + 12x^2 - 16x$.

At maxima and minima the gradient = $\frac{dy}{dx} = 0$

$\Rightarrow 4x^3 + 12x^2 - 16x = 0 \quad \Rightarrow \quad x^3 + 3x^2 - 4x = 0 \quad \Rightarrow \quad x(x^2 + 3x - 4) = 0$

$\Rightarrow\ x(x + 4)(x - 1) = 0 \quad \Rightarrow \quad x = -4, 0$ or 1.

Second find $\dfrac{d^2y}{dx^2} = 12x^2 + 24x - 16$

When $x = -4$, $\dfrac{d^2y}{dx^2} = 12 \times 16 - 24 \times 4 - 16 = 80$, *positive* $\Rightarrow$ min at $x = -4$

When $x = 0$, $\dfrac{d^2y}{dx^2} = -16$, *negative*, $\Rightarrow$ max at $x = 0$

When $x = 1$, $\dfrac{d^2y}{dx^2} = 12 + 24 - 16 = 20$, *positive*, $\Rightarrow$ min at $x = 1$.

Third find y–values: when $x = -4, 0$ or $1 \Rightarrow y = -135, -7$ or -10

$\Rightarrow$ **Maximum** at $(0, -7)$ and **Minimum** at $(-4, -135)$ and $(1, -10)$.

N.B. If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, it does not help! You can have any of max, min or stationary point of inflection.

Using gradients before and after

Example: Find the stationary points of $y = 3x^4 - 8x^3 + 6x^2 + 7$.

Solution: $y = 3x^4 - 8x^3 + 6x^2 + 7$

$\Rightarrow \quad \dfrac{dy}{dx} = 12x^3 - 24x^2 + 12x = 0$ for stationary points

$\Rightarrow \quad x(x^2 - 2x + 1) = 0 \quad \Rightarrow \quad x(x - 1)^2 = 0 \quad \Rightarrow \quad x = 0$ or 1.

$\dfrac{d^2y}{dx^2} = 36x^2 - 48x + 12$

which is 12 (positive) when $x = 0 \Rightarrow$ minimum at (0, 7)

and which is 0 when $x = 1$, so we must look at gradients before and after.

x	=	0.9	1	1.1
$\dfrac{dy}{dx}$	=	+0.108	0	+0.132
		/	—	/

$\Rightarrow$ stationary point of inflection at (1, 2)

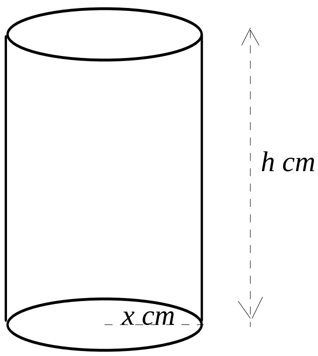
N.B. We could have *max, min or stationary point of inflection* when the second derivative is zero, so we **must** look at gradients before and after.

Maximum and minimum problems

Example:

A manufacturer of cans for baked beans wishes to use as little metal as possible in the manufacture of these cans. The cans must have a volume of 500 cm^3: how should he design the cans?

Solution:



We need to find the radius and height needed to make cans of volume 500 cm^3 using the minimum possible amount of metal.

Suppose that the radius is x cm and that the height is h cm.

The area of top and bottom together is $2 \times \pi x^2$ cm^2 and the area of the curved surface is $2\pi x h$ cm^2

$\Rightarrow$ the total surface area $A = 2\pi x^2 + 2\pi x h$ cm^2. (I)

We have a problem here: A is a function not only of x, but also of h.

But we know that the volume is 500 cm^3 and that the volume can also be written as $\pi x^2 h$ cm^3

$$\Rightarrow \quad \pi x^2 h = 500 \quad \Rightarrow \quad h = \frac{500}{\pi x^2}$$

and so (I) can be written $A = 2\pi x^2 + 2\pi x \times \dfrac{500}{\pi x^2}$

$$\Rightarrow \quad A = 2\pi x^2 + \frac{1000}{x} \quad = \quad 2\pi x^2 + 1000\,x^{-1}$$

$$\Rightarrow \quad \frac{dA}{dx} = 4\pi x - 1000x^{-2} = 4\pi x - \frac{1000}{x^2}.$$

For stationary values of A, the area, $\dfrac{dA}{dx} = 0 \;\Rightarrow\; 4\pi x = \dfrac{1000}{x^2}$

$$\Rightarrow \quad 4\pi x^3 = 1000 \quad \Rightarrow \quad x^3 = \frac{1000}{4\pi} = 79.57747155 \quad \Rightarrow \quad x = 4.301270069$$

$$\Rightarrow \quad x = 4.30 \text{ to 3 S.F.} \quad \Rightarrow \quad h = \frac{500}{\pi x^2} = 8.60$$

We do not know whether this value gives a maximum or a minimum value of A or a stationary point of inflection

so we must find $\dfrac{d^2A}{dx^2} = 4\pi + 2000x^{-3} = 4\pi + \dfrac{2000}{x^3}.$

Clearly this is positive when $x = 4.30$ and thus this gives a *minimum* of A

$\Rightarrow$ minimum area of metal is 349 cm^2
when the radius is 4.30 cm and the height is 8.60 cm.

7 Integration

Definite integrals

When limits of integration are given.

Example: Find $\int_1^3 6x^2 - 8x + 1 \;\; dx$

Solution: $\int_1^3 6x^2 - 8x + 1 \;\; dx \;\; = \left[2x^3 - 4x^2 + x\right]_1^3$ no need for $+C$ as it cancels out

$= [2 \times 3^3 - 4 \times 3^2 + 3] - [2 \times 1^3 - 4 \times 1^2 + 1]$ put top limit in first

$= [21] - [-1] = 22.$

Area under curve

The integral is the area between the curve and the x–axis, **but** areas **above** the axis are **positive and** areas **below** the axis are **negative**.

Example: Find the area between the x–axis, $x = 0$, $x = 2$ and $y = x^2 - 4x$.

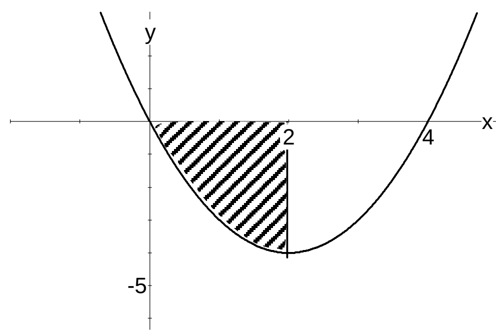
Solution:

$\int_0^2 x^2 - 4x \;\; dx$

$= \left[\dfrac{x^3}{3} - 2x^2\right]_0^2 = \left[\tfrac{8}{3} - 8\right] - [0 - 0]$

$= \dfrac{-16}{3}$ which is negative since the area is below the x–axis

$\Rightarrow$ required area is $\dfrac{+16}{3}$



Example: Find the area between the x-axis, $x = 1$, $x = 4$ and $y = 3x - x^2$.

Solution: First sketch the curve to see which bits are above (positive) and which bits are below (negative).

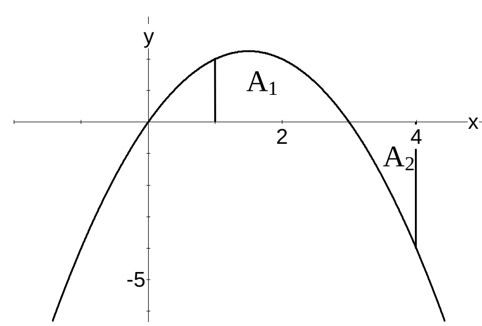
$y = 3x - x^2$

$= x(3 - x)$

$\Rightarrow$ meets x-axis at 0 and 3

so graph is as shown.

Area A_1, between 1 and 3, is above axis: area A_2, between 3 and 4, is below axis so we must find these areas separately.



$$A_1 = \int_1^3 3x - x^2 \ \ dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^3 = [4.5] - [1\tfrac{1}{6}] = 3^1/_3.$$

and $\int_3^4 3x - x^2 \ \ dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_3^4 = [2\tfrac{2}{3}] - [4.5] = -1\tfrac{5}{6}$

and so area A_2 (areas are positive) $= +1^5/_6$

so total area $= A_1 + A_2 = 3^1/_3 + 1^5/_6 = 5^1/_6$.

Note that $\int_1^4 3x - x^2 \ \ dx \ \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^4 = [2\tfrac{2}{3}] - [1\tfrac{1}{6}] = 1\tfrac{1}{2}$

which is $A_1 - A_2 = 3^1/_3 - 1^5/_6 = 1^1/_2$.

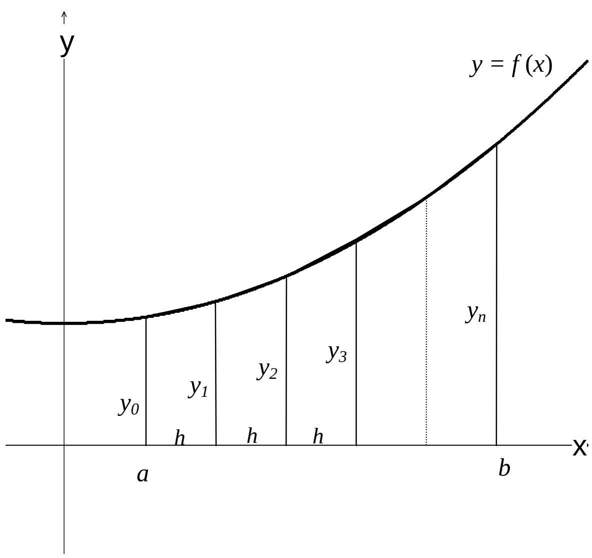
Numerical integration: the trapezium rule

Many functions can **not** be ‘anti–differentiated’ and the trapezium rule is a way of estimating the area under the curve.

Divide the area under $y = f(x)$ into n strips, each of width h.

Join the top of each strip with a straight line to form a trapezium.

Then the area under the curve
≈ sum of the areas of the trapezia



$$\Rightarrow \quad \int_a^b f(x) \ \ dx \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \tfrac{1}{2}h(y_2 + y_3) + \ \ldots \ + \tfrac{1}{2}h(y_{n-1} + y_n)$$

$$\Rightarrow \quad \int_a^b f(x) \ \ dx \approx \tfrac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \ \ldots \ + y_{n-1} + y_n)$$

$$\Rightarrow \quad \int_a^b f(x) \ \ dx \approx \tfrac{1}{2}h(y_0 + y_n + 2(y_1 + y_2 + y_3 + \ \ldots \ + y_{n-1}))$$

⇒ area under curve ≈ ½ width of each strip × (‘ends’ + 2 × ‘middles’).

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