January 2005 - Core 2

1 $y = x + \frac{2}{x}$

a) (i) $y = x + 2x^{-1}$

$\frac{dy}{dx} = 1 - 2x^{-2}$

$= 1 - \frac{2}{x^2}$

(ii) at $x = 2$, $\frac{dy}{dx} = 1 - \frac{2}{(2)^2}$

$= 1 - \frac{2}{4}$

$= 1 - \frac{1}{2} = \frac{1}{2}$ as required

b) Gradient of the tangent at $x = 2$, is $\frac{1}{2}$

$\therefore$ Gradient of the normal at $x = 2$, is -2 since $m_1 m_2 = -1$

$y = mx + c$

When $x = 2$, $y = 2 + \frac{2}{2}$

$= 2 + 1 = 3$

$3 = -2(2) + c$

$3 = -4 + c$, $c = 7$

$y = -2x + 7$

OR

$y - y_1 = m(x - x_1)$

When $x = 2$, $y = 2 + \frac{2}{2}$

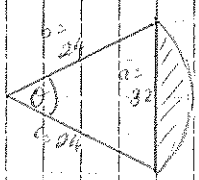
$= 2 + 1 = 3$

$y - 3 = -2(x - 2)$

$y - 3 = -2x + 4$

$y = -2x + 7$

2. Cosine Rule



a) $a^2 = b^2 + c^2 - 2bc\cos A$

$32^2 = 24^2 + 24^2 - 2(24)(24)\cos A$

$1024 = 576 + 576 - 1152 \times \cos A$

$1024 = 1152 - 1152 \times \cos A$

-1152

$-128 = -1152 \times \cos A$

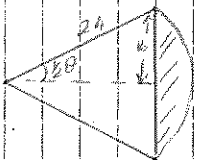
$\div -1152$

$\cos A = \frac{1}{9}$

$A = \cos^{-1}\frac{1}{9} = 1.4594\ldots$

$= 1.46$ to 3sf

OR



sin Angle = $\frac{opp}{hyp}$

$\sin\left(\frac{1}{2}\theta\right) = \frac{16}{24}$

$\frac{1}{2}\theta = \sin^{-1}\frac{16}{24}$

$\frac{1}{2}\theta = 0.7297\ldots$

$\theta = 1.4594\ldots$ or $\theta = 1.46$ to 3sf

2. b) Arc $= r\theta$

$= 24 \times 1.46$

$= 35.0269\ldots$

$= 35$ cm to the nearest cm

c) (i) Area $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 24^2 \times 1.46$

$= \frac{1}{2} \times 576 \times 1.46$

$= 420.32313$

$= 420$ cm^2 to the nearest cm^2

(ii) Segment = Area of sector − Area of triangle.

Area of triangle $= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 24 \times 24 \times \sin 1.46$

$= 286.2167\ldots$

Segment $= 420 - 286$

$= 134$ cm^2 to the nearest cm^2

3. a) from formula booklet

$u_n = a + (n-1)d$

5th term,

$u_5 = a + (5-1)d$
$= a + 4d = 46$

20th term,

$u_{20} = a + (20-1)d$
$= a + 19d = 181$

(i)

$$\begin{aligned} a + 19d &= 181 \\ a + 4d &= 46 \\ \hline 15d &= 135 \end{aligned}$$

$d = 9$ as required

(ii) $a + 4d = 46$

$a + 4(9) = 46$

$a + 36 = 46$

$a = 10$

3. b) $S_n = \frac{1}{2}n(a+l)$

or $S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$ from formula booklet

$n = 20 \quad a = 10 \quad d = 9$

$S_{20} = \frac{1}{2} \times 20 \times [2(10) + (20-1)9]$

$= 10 \times [20 + 19 \times 9]$

$= 10 \times [20 + 171]$

$= 1910$

c) $\sum_{n=21}^{50} u_n = \sum_{n=1}^{50} u_n - \sum_{n=1}^{20} u_n$

$= 11525 - 1910$

$= 9615$

4. a) $\sqrt{x} = x^{1/2}$

b) $\sqrt{x}(x-1)$

$= x^{1/2}(x-1)$

$= x^{3/2} - x^{1/2} \qquad p = 3/2$ and $q = 1/2$

c) $\int \sqrt{x}(x-1)\,dx$

$= \int x^{3/2} - x^{1/2}\,dx$

$\cancel{= \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}$

$= \frac{x^{2.5}}{2.5} - \frac{x^{1.5}}{1.5} + c$

d) $\int_1^2 \sqrt{x}(x-1)$

$= \int_1^2 x^{3/2} - x^{1/2}\,dx$

$= \left[\frac{x^{2.5}}{2.5} - \frac{x^{1.5}}{1.5}\right]_1^2$

$= \left[\frac{2^{2.5}}{2.5} - \frac{2^{1.5}}{1.5}\right] - \left[\frac{1^{2.5}}{2.5} - \frac{1^{1.5}}{1.5}\right]$

$= \left[\frac{2^2\sqrt{2}}{2.5} - \frac{2\sqrt{2}}{1.5}\right] - \left[\frac{1}{2.5} - \frac{1}{1.5}\right] = \left[6 \times \frac{4\sqrt{2}}{15} - 10 \times \frac{2\sqrt{2}}{15}\right] - \left[-\frac{4}{15}\right] = \frac{4}{15}\sqrt{2} + \frac{4}{15}$

$= \frac{4}{15}(\sqrt{2}+1)$ as required

5 a) $\log_a x = 3\log_a 6 - \log_a 8$

$\log_a x = \log_a 6^3 - \log_a 8$

$\log_a x = \log_a \left(\frac{6^3}{8}\right)$

$\log_a x = \log_a \left(\frac{216}{8}\right)$

$x = \frac{216}{8} = 27$ as required

b) (i) $\log_4 1$ $\quad 4^x = 1, \quad x = 0$

(ii) $\log_4 4$ $\quad 4^x = 4, \quad x = 1$

(iii) $\log_4 2$ $\quad 4^x = 2, \quad x = 1/2$

(iv) $\log_4 8$ $\quad 4^x = 8, \quad x = 3/2 \quad (\sqrt{4} = 2 \quad 2^3 = 8)$

6 a) (i) $(2+x)^3$

$\binom{3}{0}(2)^3(x)^0 + \binom{3}{1}(2)^2(x)^1 + \binom{3}{2}(2)^1(x)^2 + \binom{3}{3}(2)^0(x^3)$

$= 1 \times 8 \times 1 + 3 \times 4 \times x + 3 \times 2 \times x^2 + 1 \times 1 \times x^3$

$= 8 + 12x + 6x^2 + x^3$

OR

${}^3C_0 \times (2)^3 \times (x)^0 = 1 \times 8 \times 1 = 8$ +

${}^3C_1 \times (2)^2 \times (x)^1 = 3 \times 4 \times x = 12x$ +

${}^3C_2 \times (2)^1 \times (x)^2 = 3 \times 2 \times x^2 = 6x^2$ +

${}^3C_3 \times (2)^0 \times (x)^3 = 1 \times 1 \times x^3 = x^3$

$= 8 + 12x + 6x^2 + x^3$

b) $(2+x)^3 = 8 + 12x + 6x^2 + x^3$ $\qquad$ a) (ii) $8 - 12x + 6x^2 - x^3$

$(2-x)^3 = 8 - 12x + 6x^2 - x^3$

$(2+x)^3 - (2-x)^3 = 24x + 2x^3$

as required

c) $y = (2+x)^3 - (2-x)^3$

$= 24x + 2x^3$

$\frac{dy}{dx} = 24 + 6x^2$

$24 + 6x^2 = 0$

$6x^2 = -24$

$x^2 = -4$

At a stationary point, $\frac{dy}{dx} = 0$

NO REAL ROOTS.

7. a) $y = \cos 2x$

$\underline{A}$ $x = 0$

$y = \cos(2 \times 0)$
$= \cos 0$
$= 1$

$(0, 1)$

$\underline{B}$ $y = 0$

$0 = \cos 2x$
$\cos^{-1} 0 = 2x$
$90 = 2x$
$x = 45$ $(45, 0)$

$\underline{C}$ $y = -1$

$-1 = \cos 2x$
$\cos^{-1} -1 = 2x$
$180 = 2x, \quad x = 90$
$+360$
$540 = 2x, \quad \underline{x = 270}$ $(270, -1)$

by inspection

b) $y = \cos x \rightarrow y = \cos 2x$

Stretch of scale factor $\frac{1}{2}$
in the x-direction

$\boxed{\begin{array}{c} y = f(x) \Rightarrow y = f(ax) \\ \text{Stretch in the } x\text{-direction} \\ \text{of scale factor } \frac{1}{a} \end{array}}$

c) $\cos 2x = 0.37$ $\quad 0 \le x \le 360$

$\cos \theta = 0.37$ $\quad 0 \le \theta \le 720$

$\theta = 68.284\ldots$ $\quad 2x = 68.284$
$x = 34.1°$ to 1dp

or $\theta = 360 - 68.284\ldots$
$= 291.7156\ldots$ $\quad 2x = 291.716$
$x = 145.9°$ to 1dp

or $\theta = 360 + 68.284$
$= 428.2843\ldots$ $\quad 2x = 428.284$
$x = 214.1°$ to 1dp

or $\theta = 720 - 68.284$
$= 651.7156\ldots$ $\quad 2x = 651.716$
$x = 325.9°$ to 1dp

8\. a) $y = 3^x + 1$

At A, $x = 0$

$y = 3^0 + 1$
$= 1 + 1 = 2$

b) (i) $\int_0^1 (3^x + 1)$

x	0	0.25	0.5	0.75	1
$y = 3^x + 1$	2	2.316	2.732	3.279	4

from formulae booklet, Area $= \frac{1}{2} h \{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$ where $h = \frac{b-a}{n-1}$

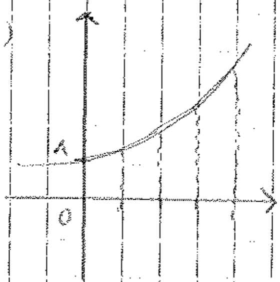
Area $= \frac{1}{2} \times \frac{1}{4} \times \{2 + 4 + 2(2.316 + 2.732 + 3.279)\}$ $\quad h = \frac{1-0}{5-1} = \frac{1}{4}$

$= \frac{1}{8} \times \{2 + 4 + 2(8.327)\}$

$= \frac{1}{8} \times \{22.654\}$

$= 2.83175$

$= 2.83$ to 3sf.

(ii) 

This is an overestimate since the trapezium used are bigger than the curve.

c) Intersection $y = 5$ and $y = 3^x + 1$

at $3^x + 1 = 5$
$3^x = 4$

$\log 3^x = \log 4$

$x \log 3 = \log 4$

$x = \frac{\log 4}{\log 3}$

$= 1.26185\dots$
$= 1.2619$ to 4dp

d) $y = 3^x + 1$ reflected in the y-axis means x-coordinates change

$f(x) = 3^{-x} + 1$

$y = f(x) \Rightarrow y = f(-x)$
reflection in the y-axis